Lecture 27

Topic

Matrix operations

Relevance

Linear operators and vectors can be represented by matrices. A knowledge of matrix analysis is required.

Objective

To understand and able to perform matrix addition, subtraction, multiplication, differentiation, integration; to be able to calculate the transpose, comple-conjugate, Hermitian conjugate matrices of a given matrix.

Addition & Subtraction of Matrices

The operation of **addition** (or **subtraction**) for two $n \times m$ matrices is defined as

$$C = A \pm B$$

where

$$c_{ij} = a_{ij} \pm b_{ij}$$

Example:

$$\begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 2 & 2 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 4 \\ 4 & 0 & 0 \end{pmatrix}$$

The commutative and associative laws are also valid for addition of matrices of the same order:

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

Equal Matrices & Null Matrix

Two matrices, A and B, of the same order are said to be **equal** if and only if $a_{ij} = b_{ij}$ for all i and j. For example,

$$A = \begin{pmatrix} 2 & 0 & -1 & 2 \\ 6 & 5 & 3 & 7 \\ 2 & -1 & 0 & 4 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 2 & 0 & -1 & 2 \\ 6 & 5 & 3 & 7 \\ 2 & -1 & 0 & 4 \end{pmatrix}$$

If $a_{ij} = 0$ for all i and j, then A is called a **null matrix**. For example

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Multiplication of a Matrix by a Scalar

Each element of the matrix is multiplied by the scalar. The elements of kA are therefore ka_{ij} for all i and j. For example

$$2\begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 & 4 \\ 4 & -6 & -2 \\ 2 & 4 & 2 \end{pmatrix}$$

Matrix Product

Matrix product is defined for conformable matrices only. This means that the number of columns of A must equal the number of rows of B for $C = AB$. Consider

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\begin{aligned} AB &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}. \end{aligned}$$

In general, the matrix product $C = AB$ is obtained by use of the following definition:

$$c_{ij} = \sum_{k=1}^{l} a_{ik} b_{kj}$$

where the orders of A, B, and C are $n \times l$, $l \times m$, and $n \times m$, respectively.

Examples of Matrix Product

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 3 \end{pmatrix}$$

Consider the following system of equations

$$\begin{aligned} 3x + y + 2z &= 3 \\ 2x - 3y - z &= -3 \\ x + 2y + z &= 4 \end{aligned}$$

In matrix form, it can be written as

$$\begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix}$$

Matrix Multiplication

Consider

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$$

It can be shown by use of the definition of the matrix product, the commutative law of multiplication is not, in general, valid for the matrix product,

$$AB \neq BA.$$

However, the associative law of multiplication is valid for the matrix product,

$$A(BC) = (AB)C$$

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Transpose Matrix: A^T

The **transpose** of an arbitrary matrix A is written as A^T and is obtained by interchanging corresponding rows and columns of A, that is, if the element of ith row and jth column of A is a_{ij}, then the element at ith row and jth column of A^T is a_{ji}.

For example

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 2 & 1 \end{pmatrix}$$

It can be shown that

$$(AB)^T = B^T A^T$$

$$(ABC \cdots G)^T = G^T \cdots C^T B^T A^T$$

Complex-Conjugate Matrix: A^*

The **complex conjugate** of an arbitrary matrix A is formed by taking the complex conjugate of each element. Hence we have

$$(A^*)_{ij} = a^*_{ij} \qquad \text{for all } i \text{ and } j.$$

For example

$$A = \begin{pmatrix} 2+3i & 4-5i \\ 3 & 4i \end{pmatrix}$$

$$A^T = \begin{pmatrix} 2-3i & 4+5i \\ 3 & -4i \end{pmatrix}$$

If $A^* = A$, then A is a **real matrix**.

Hermitian Conjugate: $A^\dagger$

The **Hermitian conjugate** (or adjoint) of an arbitrary matrix A is obtained by taking the *complex conjugate* of the matrix and then the *transpose* of the complex conjugate matrix.

$$A^\dagger = (A^*)^T = (A^T)^*$$

For example

$$A = \begin{pmatrix} 2+3i & 4-5i \\ 3 & 4i \end{pmatrix}$$

$$A^\dagger = \begin{pmatrix} 2-3i & 4+5i \\ 3 & -4i \end{pmatrix}^T = \begin{pmatrix} 2-3i & 3 \\ 4+5i & -4i \end{pmatrix}$$

It can be shown that

$$(AB)^\dagger = B^\dagger A^\dagger$$

$$(AB \cdots G)^\dagger = G^\dagger \cdots B^\dagger A^\dagger$$

Inner Product of Vectors

Assume that in a given orthonormal basis the vectors **a** and **b** are represented by the column matrices

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}$$

For real vectors

$$\mathbf{a}^T\mathbf{b} = (a_1 \ a_2 \ \cdots \ a_N) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} = \sum_{i=1}^{N} a_i b_i$$

For complex vectors

$$\mathbf{a}^\dagger\mathbf{b} = (a_1^* \ a_2^* \ \cdots \ a_N^*) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} = \sum_{i=1}^{N} a_i^* b_i$$

is the inner product $\langle \mathbf{a} | \mathbf{b} \rangle$ in that basis.