Lecture 28

Topic

Speical square matrices

Relevance

Square matrices are very common in physical applications. We will consider some special forms of square matrix that are of particular importance.

Relevance

To know the special properties of different types of square matrices.

Special Square Matrices

null matrix
identity matrix
diagonal matrix
symmetric matrix
antisymmetric matrix
orthogonal matrix
Hermitian matrix
unitary matrix
cofactor matrix
the inverse of a matrix

adjoint matrix (same as Hermitian conjugate)
self-adjoint matrix (same as Hermitian)
singular matrix: $|A| = 0$

Null & Identity Matrices

The null or zero matrix 0 has all elements equal to zero, and the properties

$$A0 = 0 = 0A$$

$$A + 0 = 0 + A = A$$

The **identity (unit) matrix** is defined by

$$IA = AI = A$$

and is given by

$$(I)_{ij} = \delta_{ij} = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases}$$

where δ_{ij} is the **Kronecker delta** function.

The 3×3 identity matrix is given by

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Diagonal Matrix

A diagonal matrix has non-zero elements only on the leading diagonal. All off-diagonal elements are zero.

For example

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

is a 3×3 diagonal matrix.

Symmetric & Antisymmetric Matrix

A square matrix A is said to be symmetric if

$$A^T = A$$

A square matrix A is said to be antisymmetric if

$$A^T = -A$$

For example, σ_1 below is symmetric, σ_2 is antisymmetric.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_1^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_2^T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -\sigma_2$$

The diagonal elements of an antisymmetric matrix are necessarily zero.

Orthogonal Matrix

If a real matrix A has the property

$$AA^T = I \quad \text{or} \quad A^T = A^{-1}$$

A is said to be an **orthogonal matrix**.

For example,

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad R^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$R^T R = I$$

An orthogonal matrix represents (in a particular basis) a linear operator that leaves the norms (lengths) of vectors unchanged.

$$\mathbf{x}' = A\mathbf{x}$$

$$(\mathbf{x}')^T\mathbf{x}' = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T\mathbf{x}$$

i.e.

$$\langle \mathbf{x}' | \mathbf{x}' \rangle = \langle \mathbf{x} | \mathbf{x} \rangle$$

Hermitian Matrix

If

$$A^\dagger = A,$$

A is said to be a **Hermitian matrix**.

For example,

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_2^* = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$\sigma_2^\dagger = (\sigma_2^*)^T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2$$

Unitary Matrix

A matrix A is said to be a **unitary matrix**, if

$$A^\dagger A = I, \quad \text{, or} \quad A^\dagger = A^{-1}$$

For example,

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_2^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_2^\dagger \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

An unitary matrix represents a linear operator that leaves the norm of a (complex) vector unchanged. If

$$\mathbf{x}' = A\mathbf{x}$$

$$(\mathbf{x}')^\dagger \mathbf{x}' = (A\mathbf{x})^\dagger (A\mathbf{x}) = \mathbf{x}^\dagger A^\dagger A \mathbf{x} = \mathbf{x}^\dagger \mathbf{x}$$

i.e.

$$\langle \mathbf{x}' | \mathbf{x}' \rangle = \langle \mathbf{x} | \mathbf{x} \rangle$$

Minor and Cofactor

The **minor** M_{ij} of the element a_{ij} of an $N \times N$ matrix A is the determinant of the $(N-1) \times (N-1)$ matrix obtained by removing all the elements of the ith row and jth column of A. For example

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The **cofactor** A^{ij} of the element a_{ij} is found by multiplying the minor by $(-1)^{i+j}$. For the matrix above, the cofactor of a_{13} is

$$A^{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Similarly, the minor and cofactor of a_{32} are

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \qquad A^{23} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Cofactor Matrix

The cofactor matrix A^c of matrix A is obtained by replacing every element by its cofactor. For example

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$A^c = \begin{pmatrix} A^{11} & A^{12} & A^{13} \\ A^{21} & A^{22} & A^{23} \\ A^{31} & A^{32} & A^{33} \end{pmatrix}$$

Example

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$$

$$A^c = \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}$$