Lecture 29

Topic

The inverse of a matrix

Relevance

The inverse of a matrix is very important in physical applications such as solving liear system of equations, in transformation of operators between different basis, etc.

Aim

To understand the concept of inverse of a matrix and able to compute the inverse of a given matrix.

The Inverse of a Matrix

For numbers and functions, if $ab = 1$, then $b = 1/a = a^{-1}$, such that

$$aa^{-1} = 1$$

The **inverse** of a matrix (if it has one) is the matrix A^{-1} such that

$$AA^{-1} = I = A^{-1}A$$

For numbers and functions, if $c = ab$, then

$$a = \frac{c}{b} = cb^{-1} \quad \text{and} \quad b = \frac{c}{a} = ca^{-1}$$

For matrices, if $P = AB$, then

$$B = A^{-1}P$$

$$A = PB^{-1}$$

Furthermore

$$A^{-1}AB = B$$

Finding the Inverse of a Matrix

Consider

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

Assume that

$$A^{-1} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

Then

$$\begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The left hand side of the above equation can be written as

$$\begin{pmatrix} x_1 - z_1 & x_2 - z_2 & x_3 - z_3 \\ -2x_1 + y_1 & -2x_2 + y_2 & -2x_3 + y_3 \\ x_1 - y_1 + 2z_1 & x_2 - y_2 + 2z_2 & x_3 - y_3 + 2z_3 \end{pmatrix}$$

Finding the Inverse of a Matrix (cont.)

Compare this with the right hand side, we get

$$\begin{cases} x_1 & & -z_1 & = 1 \\ -2x_1 & +y_1 & & = 0 \\ x_1 & -y_1 & +2z_1 & = 0 \end{cases}$$

$$\begin{cases} x_2 & & -z_2 & = 0 \\ -2x_2 & +y_2 & & = 1 \\ x_2 & -y_2 & +2z_2 & = 0 \end{cases}$$

$$\begin{cases} x_3 & & -z_3 & = 0 \\ -2x_3 & +y_3 & & = 0 \\ x_3 & -y_3 & +2z_3 & = 1 \end{cases}$$

Solving these linear systems of equations, we have

$$\begin{array}{lll} x_1 = 2 & y_1 = 4 & z_1 = 1 \\ x_2 = 1 & y_2 = 3 & z_2 = 1 \\ x_3 = 1 & y_3 = 2 & z_3 = 1 \end{array}$$

$$A^{-1} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

It can be verified that $AA^{-1} = I = A^{-1}A$ (do it!).

Finding the Inverse of a Matrix

It can be shown that

$$A^{-1} = \frac{(A^c)^T}{|A|}$$

where A^c is the cofactor matrix of A and $|A|$ is the determinant of A.

Example: Find the inverse of

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

$$A^{11} = (-1)^{1+1} \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2$$

$$A^{12} = (-1)^{1+2} \begin{vmatrix} -2 & 0 \\ 1 & 2 \end{vmatrix} = 4$$

$$A^{13} = (-1)^{1+3} \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = 1$$

$$A^{21} = (-1)^{2+1} \begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix} = 1$$

Finding the Inverse of a Matrix

$$A^{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 3$$

$$A^{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1$$

$$A^{31} = (-1)^{3+1} \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

$$A^{32} = (-1)^{3+2} \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 2$$

$$A^{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1$$

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix} = 1$$

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} A^{11} & A^{21} & A^{31} \\ A^{12} & A^{22} & A^{32} \\ A^{13} & A^{23} & A^{33} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$