Lecture 30

Topic

Systems of linear equations

Relevance

For a chosen set of basis in a linear space, vectors and operators are represented by matrices. Many problems in physics thus lead to the solution of a system of linear equations in several unkowns. Solving systems of linear equations is therefore very important.

Aims

To be able to solve systems of linear equations using

- simple substitution
- Gaussian elimination with back substitution
- Cramer's rule

Systems of Linear Equations

Simple systems of equations can be solved by substitutions or elimination.

$$\begin{cases} 2x & & -z & = 2 \\ 6x & +5y & +3z & = 7 \\ 2x & -y & & = 4 \end{cases}$$

From the first and the third equations, we get

$$z = 2x - 2 \qquad y = 2x - 4$$

Substitute these into the second equation, we have

$$6x + 5(2x - 4) + 3(2x - 2) = 7$$

or

$$22x = 33$$

$$x = 3/2$$

The values of y and z can then be found,

$$y = -1 \qquad z = 1$$

Gaussian Elimination: An Example

For more complicated problems, we need a *systematic* method of solution. **Gaussian elemination** or **row reduction** method is useful for numerical computation and leads to efficient methods of solution of large sets of equations by computer.

Consider

$$\begin{cases} 2x & & -z & = 2 \\ 6x & +5y & +3z & = 7 \\ 2x & -y & & = 4 \end{cases}$$

If we always write the equations in this *standard form*, the unknowns x, y and z can be omitted and the equations can be written as

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 6 & 5 & 3 & 7 \\ 2 & -1 & 0 & 4 \end{array}\right)$$

Example (cont.)

$$\left\{\begin{array}{llll} 2x & & -z & = 2 \\ 6x & +5y & +3z & = 7 \\ 2x & -y & & = 4 \end{array}\right. \qquad \left(\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 6 & 5 & 3 & 7 \\ 2 & -1 & 0 & 4 \end{array}\right)$$

Subtract the first equation from the third

$$\left\{\begin{array}{llll} 2x & & -z & = 2 \\ 6x & +5y & +3z & = 7 \\ & -y & +z & = 2 \end{array}\right. \qquad \left(\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 6 & 5 & 3 & 7 \\ 0 & -1 & 1 & 2 \end{array}\right)$$

Multiply the 1st equation by -3 and add to the 2nd,

$$\left\{\begin{array}{llll} 2x & & -z & = 2 \\ & 5y & +6z & = 1 \\ & -y & +z & = 2 \end{array}\right. \qquad \left(\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 5 & 6 & 1 \\ 0 & -1 & 1 & 2 \end{array}\right)$$

Exchange the second and the third equations

$$\left\{\begin{array}{llll} 2x & & -z & = 2 \\ & -y & +z & = 2 \\ & 5y & +6z & = 1 \end{array}\right. \qquad \left(\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 5 & 6 & 1 \end{array}\right)$$

Example (cont.)

Multiply the 2rd equation by 5, add to the 3rd,

$$\begin{cases} 2x & & -z & = 2 \\ & -y & +z & = 2 \\ & & 11z & = 11 \end{cases} \qquad \left(\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 11 & 11 \end{array}\right)$$

The 3rd equation now gives

$$z = 1$$

Substitute $z = 1$ to the 2nd equation, y can be found,

$$-y + 1 = 2 \quad \longrightarrow \quad y = -1$$

Finally substitute $y = -1$ and $z = 1$ back to the first equation, we found

$$2x - 1 = 2 \quad \longrightarrow \quad x = \frac{3}{2}$$

The solution of the system of linear equations is therefore

$$(x, y, z) = \left(\frac{3}{2}, -1, 1\right)$$

Gaussian Elimination

The solution can be obtained by performing *elementary row operations* on the matrix. Each step of solving the equations (multiplying an equation by a constant, adding one equation to another, etc) corresponds to an elementary row operation.

Allowed row operations include:

- Interchange two rows;
- Multiply (or divide) a row by a (nonzero) constant;
- Add (or subtract) a multiple of one row to (from) another.

The *aim* is to make the lower left coner of the first part (square) of the matrix zero.

Summary of Procedure

1. Be sure the equations are in standard form (x terms lined up, etc., constants on right-hand side) and write the corresponding matrix.

2. If necessary, interchange or combine rows to obtain a suitable *pivot* in the upper left-hand corner of the matrix. To avoid fractions, try to obtain a pivot which divides evenly into the numbers below it.

3. Use the pivot and elementary row operations to make all numbers below the pivot zeros.

4. Now ignore the first row and first column of the matrix and repeat steps 2 and 3 on the remaining submatrix. If there are more than three rows, next ignore the first two rows and first two columns, and so on until the matrix is in *echelon* form.

5. Starting at the bottom, back substitute to find the solution.

Example

$$\begin{cases} & 10y & -z & +w & = & 10 \\ 2x & -2y & -4z & & = & -3 \\ 4x & +2y & & +4w & = & 5 \\ 3x & +2y & & +3w & = & 4 \end{cases}$$

Write in matrix form,

$$\left(\begin{array}{cccc|c} 0 & 10 & -1 & 1 & 10 \\ 2 & -2 & -4 & 0 & -3 \\ 4 & 2 & 0 & 4 & 5 \\ 3 & 2 & 0 & 3 & 4 \end{array}\right)$$

$$R3 - R4 \implies \left(\begin{array}{cccc|c} 0 & 10 & -1 & 1 & 10 \\ 2 & -2 & -4 & 0 & -3 \\ 1 & 0 & 0 & 1 & 1 \\ 3 & 2 & 0 & 3 & 4 \end{array}\right)$$

$$R1 \leftrightarrow R3 \implies \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 2 & -2 & -4 & 0 & -3 \\ 0 & 10 & -1 & 1 & 10 \\ 3 & 2 & 0 & 3 & 4 \end{array}\right)$$

Example (cont.)

$$R2 - 2 \times R1 \quad R4 - 3 \times R1 \implies \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & -2 & -4 & -2 & -5 \\ 0 & 10 & -1 & 1 & 10 \\ 0 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$R2 \leftrightarrow R4 \implies \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 10 & -1 & 1 & 10 \\ 0 & -2 & -4 & -2 & -5 \end{array}\right)$$

$$R3 - 5 \times R2 \quad R4 + R2 \implies \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & -4 & -2 & -4 \end{array}\right)$$

$$R4 - 4 \times R3 \implies \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -6 & -24 \end{array}\right)$$

Example (cont.)

The corresponding equations become

$$\begin{cases} x & & & +w & = & 1 \\ & 2y & & & = & 1 \\ & & -z & +w & = & 5 \\ & & & -6w & = & -24 \end{cases}$$

Perform back substitution: From the last equation

$$-6w = -24 \quad \longrightarrow \quad w = 4$$

Substitute $w = 4$ into the 3rd equation

$$-z + 4 = 5 \quad \longrightarrow \quad z = -1$$

From the 2nd equation

$$2y = 1 \quad \longrightarrow \quad y = 1/2$$

Finally from the 1st equation

$$x + w = 1 \quad \longrightarrow \quad x = -3$$

The solution is then

$$(x, y, z, w) = \left(-3, \frac{1}{2}, -1, 4\right)$$

Cramer's Rule

For a system of linear equations with numerical coefficients, row reduction involves less arithmetic. However, if the coefficients are not numerical, and for theoretical purposes where a formula rather than a numerical answer is wanted, row reduction is not practical, and **Cramer's rule** is very useful.

Consider

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

To eliminate y, we multiply the first equation by b_2, the second equation by b_1, and then subtract the two equations

$$(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1$$

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

Solving for y in a similar way, we get

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

Cramer's Rule

The above can be written as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Thise gives the solution of

$$\begin{cases} a_1 x & + & b_1 y & = & c_1 \\ a_2 x & + & b_2 y & = & c_2 \end{cases}$$

To find the correct determinants, the equations must be written in the standard form as for row reduction. Then the denominator determinant in the solution is formed by the array of coefficients on the left-hand side of the equations, This determinant is called the *determinant of the coefficients* and is denoted by D. To find the numerator determinant for x, start with D, erase the x coefficients a_1 and a_2, and replace them by the constants c_1 and c_2 from the right-hand sides of the equations. Similarly, we replace the y coefficients in D by the constant terms to find the numerator determinant in y.

Cramer's Rule

The Cramer's rule can be used to solve n equations in n unknowns if $D \neq 0$; the solution then consists of one value for each unknown. The denominator determinant D is the $n \times n$ determinant of the coefficients when the equations are arranged in standard form. The numerator determinant for each unknown is the determinant obtained by replacing the column of coeffients of that unknown in D by the constant terms from the right-hand sides of the equations. Then to find the unknowns, we must evaluate each of the determinants and divide.

Cramer's Rule

$$AX = B$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Solution:

$$x_j = \frac{D_j}{D}$$

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

$$D_j = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$