Lecture 31

Topic:

Calculating determinants

Relevance:

Evaluation of determinants is crucial in solving simultaneous linear equations using the Cramer's rule, in finding the inverse of a matrix, in eigenvalue problems, and many other applications.

Aim:

- To understand the properties of determinant.
- To be able to evaluate determinants using Laplace's development.

Calculating Determinant

To find the value of a determinant, we multiply each element of one row (or one column) by its cofactor and add the results. It can be shown that we get the same answer whichever row or column we use.

$$\begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} = 3 \times 5 - 4 \times 1 = 11$$

$$\begin{aligned} \begin{vmatrix} 1 & -5 & 2 \\ 7 & 3 & 4 \\ 2 & 1 & 5 \end{vmatrix} &= 1 \times (-1)^{1+1} \begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} \\ & \quad -5 \times (-1)^{1+2} \begin{vmatrix} 7 & 4 \\ 2 & 5 \end{vmatrix} + 2 \times (-1)^{1+3} \begin{vmatrix} 7 & 3 \\ 2 & 1 \end{vmatrix} \\ &= 11 + 135 + 2 \\ &= 148 \end{aligned}$$

Laplace's Development

The above method of evaluating a determinant is referred as Laplace's development of a determinant. If the determinant is of fourth order (or higher), using the Laplace's development once gives us a set of determinants of order one less than we started with; then we use the Laplace development all over again to evaluate each of these, and so on until we get determinants of second order which we know how to evaluate.

The calculation can be simplified. The idea is to use the properties of determinant to get as many zeros as possible in some row or column in order to have fewer terms in the Lapalace development. This can be done efficiently by a method very similar to the process of reducing a matrix to row echelon form.

Properties of Determimant

1. The value of a determinant is not changed if corresponding rows and columns are interchanged.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. If a multiple of one column is added (row by row) to another column or if a multiple of one row is added (column by column) to another row, the value of the determinant is unchanged.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + kb_1 & b_1 & c_1 \\ a_2 + kb_2 & b_2 & c_2 \\ a_3 + kb_3 & b_3 & c_3 \end{vmatrix}$$

Properties of Determimant (cont.)

3. If each element of a column or row is zero, the value of the determinant is zero.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

4. If two columns or rows are identical, the value of the determinant is zero.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$$

5. If two columns or rows are proportional, the vlue of the determinant is zero.

$$\begin{vmatrix} 2 & 4 & 5 \\ 3 & 6 & -2 \\ 1 & 2 & 7 \end{vmatrix} = 0$$

Properties of Determimant (cont.)

6. If two columns or rows are interchanged, the sign of the determinant is changed.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}$$

7. If each element of a column or row is multiplied by the same number, the resulting determinant is multiplied by that same number.

$$\begin{vmatrix} a_1 & b_1 & kc_1 \\ a_2 & b_2 & kc_2 \\ a_3 & b_3 & kc_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

8. If A, B, $\cdots$ and G are square matrices of the same order, then

$$|AB| = |A||B| = |BA|$$

$$|AB \cdots G| = |A||B| \cdots |G| = |B \cdots GA|$$

Example

Evaluate the determinant D:

$$D = \begin{vmatrix} 4 & 3 & 0 & 1 \\ 9 & 7 & 2 & 3 \\ 4 & 0 & 2 & 1 \\ 3 & -1 & 4 & 0 \end{vmatrix}$$

Interchange column 1 and column 4 (note minus sign)

$$D = -\begin{vmatrix} 1 & 3 & 0 & 4 \\ 3 & 7 & 2 & 9 \\ 1 & 0 & 2 & 4 \\ 0 & -1 & 4 & 3 \end{vmatrix}$$

Factor 2 out of column 3

$$D = -2\begin{vmatrix} 1 & 3 & 0 & 4 \\ 3 & 7 & 1 & 9 \\ 1 & 0 & 1 & 4 \\ 0 & -1 & 2 & 3 \end{vmatrix}$$

Example (cont.)

Row reduction: $R2 - 3 \times R1$

$$D = -2 \begin{vmatrix} 1 & 3 & 0 & 4 \\ 0 & -2 & 1 & -3 \\ 1 & 0 & 1 & 4 \\ 0 & -1 & 2 & 3 \end{vmatrix}$$

Row reduction: $R3 - R1$

$$D = -2 \begin{vmatrix} 1 & 3 & 0 & 4 \\ 0 & -2 & 1 & -3 \\ 0 & -3 & 1 & 0 \\ 0 & -1 & 2 & 3 \end{vmatrix}$$

Laplace development

$$D = -2 \begin{vmatrix} -2 & 1 & -3 \\ -3 & 1 & 0 \\ -1 & 2 & 3 \end{vmatrix}$$

Example (cont.)

Interchange column 1 and column 2

$$D = 2\begin{vmatrix} 1 & -2 & -3 \\ 1 & -3 & 0 \\ 2 & -1 & 3 \end{vmatrix}$$

Row reduction: $R2 - R1$

$$D = 2\begin{vmatrix} 1 & -2 & -3 \\ 0 & -1 & 3 \\ 2 & -1 & 3 \end{vmatrix}$$

Row reduction: $R3 - 2 \times R1$

$$D = 2\begin{vmatrix} 1 & -2 & -3 \\ 0 & -1 & 3 \\ 0 & 3 & 9 \end{vmatrix}$$

Laplace development

$$D = 2\begin{vmatrix} -1 & 3 \\ 3 & 9 \end{vmatrix} = 2(-9 - 9) = -36$$