# Lecture 32

## Topic:

Eigenvalue problem

## Relevance:

Eigenvalue problem is a very common physics problem. Small oscillation of a physical system, time-independent Schrödinger equation in quantum mechanics, are good examples of eigenvalue problem.

## Aim:

- Understand the nature of eigenvalue problem.
- Given the matrix representation of an operator, to be able to find the eigenvalues and eigenvectors.
- Understand simple properties of eigenvalues and eigenvectors.

# Eigenvalue Problems

When an operator $A$ acts on a vector $|\psi\rangle$, the resulting vector $A|\psi\rangle$ is, in general, distinct from $|\psi\rangle$. However, there may exist certain (nonzero) vector for which $A|\psi\rangle$ is just $|\psi\rangle$ multiplied by a constant $\lambda$. That is

$$A|\psi\rangle = \lambda|\psi\rangle$$

Such a vector is called an **eigenvector** of the operator $A$, and the constant $\lambda$ is called an **eigenvalue**. The eigenvector is said to "belong" to the eigenvalue.

A good example of eigenvalue problem is the time independent Schrödinger equation in quantum physics,

$$H|\psi\rangle = E|\psi\rangle$$

For a single particle in a field with potential $V$,

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V$$

# Matrix Notation

In a given coordinate system, the eigenvalue problem can be expressed in matrix notation

$$A\psi = \lambda\psi$$

The $i$-component of it is

$$\sum_j A_{ij}\psi_j = \lambda\psi_i$$

The matrix equation can also be written as

$$(A - \lambda I)\psi = 0$$

The condition for a nontrival solution of the above equation is

$$|A - \lambda I| = 0$$

This is called the **secular** (or **characteristic**) equation of $A$.

The problem of finding the eigenvalues $\lambda$ for which the system of linear equations has a nontrival solution is a very important one.

# Example

If $A$ is the matrix

$$\begin{pmatrix} -3 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$(A - \lambda I)x = 0$ becomes

$$\left[ \begin{pmatrix} -3 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \left[ \begin{pmatrix} -3 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 - \lambda & 2 & 2 \\ 2 & 1 - \lambda & 3 \\ 2 & 3 & 1 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

# Example (cont.)

Condition for a nontrival solution of the above system of equations is

$$\begin{vmatrix} -3-\lambda & 2 & 2 \\ 2 & 1-\lambda & 3 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$-(3+\lambda)(1-\lambda)^2 + 24 - 8(1-\lambda) + 9(3+\lambda) = 0$$

$$(3+\lambda)\left[9-(1-\lambda)^2\right] + 8[3-(1-\lambda)] = 0$$

$$(3+\lambda)[3+(1-\lambda)][3-(1-\lambda)] + 8[3-(1-\lambda)] = 0$$

$$[3-(1-\lambda)]\{8+(3+\lambda)[3+(1-\lambda)]\} = 0$$

$$(2+\lambda)\{8+(3+\lambda)(4-\lambda)\} = 0$$

$$(2+\lambda)(-\lambda^2+\lambda+20) = 0$$

$$(2+\lambda)(4+\lambda)(5-\lambda) = 0$$

# Example (cont.)

Solution: $\lambda_1 = -2, \quad \lambda_2 = -4, \quad \lambda_3 = 5$

Since $\begin{pmatrix} -3-\lambda & 2 & 2 \\ 2 & 1-\lambda & 3 \\ 2 & 3 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

When $\lambda = -2$,

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Because $|A - \lambda I| = 0$, two of the equations are always linearly dependent. As a results, a unique solution cannot be obtained. But the $N-1$ linearly independent equations can be used to express the $N-1$ variables in terms of the remaining one. Here

$$x_2 = -x_3 \qquad x_1 = 0$$

The value of $x_3$ is determined by other requirement, such as normalization of the vector. $||x|| = 1$. Then

$$(x_1, x_2, x_3) = \frac{1}{\sqrt{2}}(0, -1, 1)$$

# Example (cont.)

For $\lambda = -4$

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 3 \\ 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 = -4x_3 \qquad x_2 = x_3$$

The normalized eigenvector is thus

$$(x_1, x_2, x_3) = \frac{1}{3\sqrt{2}}(-4, 1, 1)$$

For $\lambda = 5$

$$\begin{pmatrix} -8 & 2 & 2 \\ 2 & -4 & 3 \\ 2 & 3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 = x_3/2 \qquad x_2 = x_3$$

The normalized eigenvector is thus

$$(x_1, x_2, x_3) = \frac{1}{3}(1, 2, 2)$$

# Eigenvalue Problem

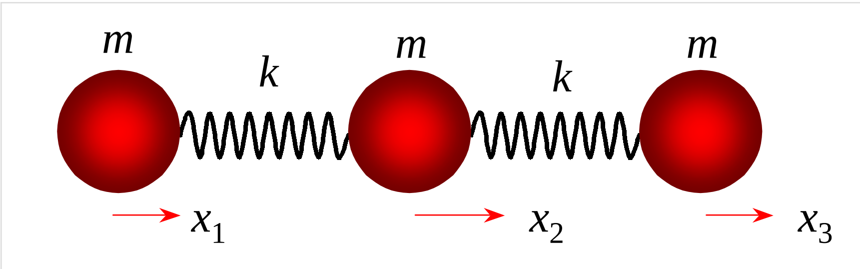
In general, the eigenvalues of a matrix $A$ are determined by the secular equation

$$|A - \lambda I| = 0$$

If $A$ is a $n \times n$ matrix, the left hand side of the above equation is an $n$th degree polynomial and the equation has *$n$ roots*, not necessarily all different. The system is then said to have $n$ eigenstates.

# Oscillatory System

Consider the mechanical oscillation of the following mass-spring system. Assume that the three particles have the same mass ($m$) and they are connected by identical springs of spring constant ($k$).




Equations of motion:

$$\begin{aligned} m\ddot{x}_1 &= k(x_2 - x_1) \\ m\ddot{x}_2 &= k(x_3 - x_2) - k(x_2 - x_1) \\ m\ddot{x}_3 &= -k(x_3 - x_2) \end{aligned}$$

Assume the solution of of the following form

$$\begin{aligned} x_1 &= A_1 \cos(\omega t + \phi_1) \quad \ddot{x}_1 = -\omega^2 x_1 \\ x_2 &= A_2 \cos(\omega t + \phi_2) \quad \ddot{x}_1 = -\omega^2 x_1 \\ x_3 &= A_3 \cos(\omega t + \phi_3) \quad \ddot{x}_1 = -\omega^2 x_1 \end{aligned}$$

# Oscillatory System (cont.)

The equations on the last slide can be written as

$$\begin{array}{rrrcl} -kx_1 & +kx_2 & & = & -m\omega^2 x_1 \\ kx_1 & -2kx_2 & +kx_3 & = & -m\omega^2 x_2 \\ & kx_2 & -kx_3 & = & -m\omega^2 x_3 \end{array}$$

or

$$\begin{pmatrix} -\omega_0^2 & \omega_0^2 & 0 \\ \omega_0^2 & -2\omega_0^2 & \omega_0^2 \\ 0 & \omega_0^2 & -\omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -\omega^2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where $\omega_0^2 = k/m$. This is an eigenvalue problem and the eigenvalue is $-\omega^2$. The secular equation is given by

$$\begin{vmatrix} \omega^2 - \omega_0^2 & \omega_0^2 & 0 \\ \omega_0^2 & \omega^2 - 2\omega_0^2 & \omega_0^2 \\ 0 & \omega_0^2 & \omega^2 - \omega_0^2 \end{vmatrix} = 0$$

or

$$\left(\omega^2 - \omega_0^2\right)^2 \left(\omega^2 - 2\omega_0^2\right) - 2\omega_0^4 \left(\omega^2 - \omega_0^2\right) = 0$$

Eigenvalues:

$$\omega_1 = 0, \quad \omega_2 = \omega_0 = \sqrt{\frac{k}{m}}, \quad \omega_3 = \sqrt{3}\omega_0 = \sqrt{\frac{3k}{m}}$$

# Oscillatory System (cont.)

Eigenvector corresponding to $\omega_1 = 0$,

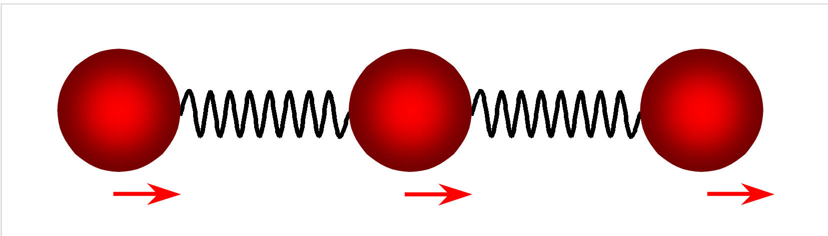
$$\begin{pmatrix} -\omega_0^2 & \omega_0^2 & 0 \\ \omega_0^2 & -2\omega_0^2 & \omega_0^2 \\ 0 & \omega_0^2 & -\omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Longrightarrow \quad x_1 = x_2 = x_3$$

or

$$x^1 = (x_1, x_2, x_3) = \frac{1}{\sqrt{3}}(1, 1, 1)$$

Normal mode:



# Oscillatory System (cont.)

Eigenvector corresponding to $\omega_2 = \omega_0 = \sqrt{k/m}$,

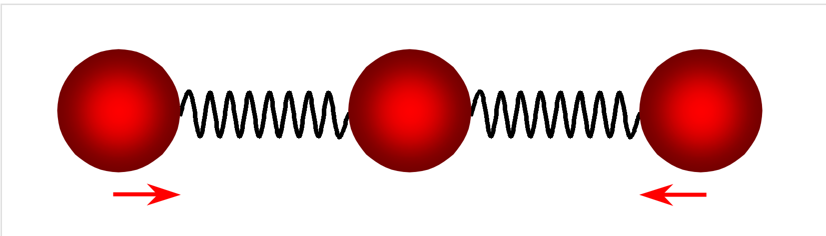
$$\begin{pmatrix} 0 & \omega_0^2 & 0 \\ \omega_0^2 & -\omega_0^2 & \omega_0^2 \\ 0 & \omega_0^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Longrightarrow \qquad x_1 = -x_3, \; x_2 = 0$$

or

$$x^2 = (x_1, x_2, x_3) = \frac{1}{\sqrt{2}}(1, 0, -1)$$

Normal mode:



# Oscillatory System (cont.)

Eigenvector corresponding to $\omega_2 = \sqrt{3}\omega_0 = \sqrt{3k/m}$,

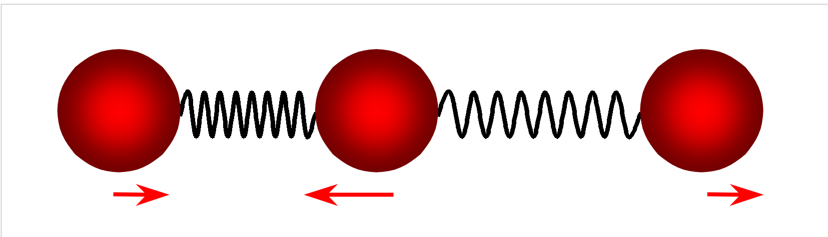
$$\begin{pmatrix} 2\omega_0^2 & \omega_0^2 & 0 \\ \omega_0^2 & \omega_0^2 & \omega_0^2 \\ 0 & \omega_0^2 & 2\omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

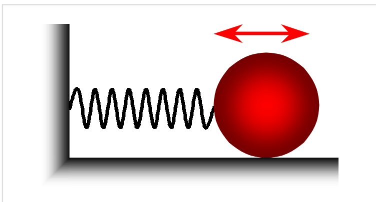
$$\Longrightarrow \quad x_1 = x_3, \; x_2 = -2x_1$$

or

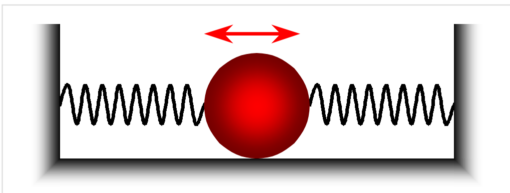
$$x^3 = (x_1, x_2, x_3) = \frac{1}{\sqrt{6}}(1, -2, 1)$$

Normal mode:





$$\omega = \sqrt{\frac{k}{m}}$$



$$\omega = \sqrt{\frac{2k}{m}}$$

# Observations

1. For mechanical oscillation, the eigenvalues are the squares of vibration frequencies. Therefore, it is required that $\omega^2 \geq 0$. A symmtric (Hermitian) matrix $A$ all eigenvalues positive is called positive definite. If it is possible that some of the eigenvalues are zero, then $A$ is called positive semi-definite. The following matrix is positive semi-definite and its eigenvalues are $0$, $\omega_0^2$ and $3\omega_0^2$.

$$\begin{pmatrix} \omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & \omega_0^2 \end{pmatrix}$$

2. In general, the eigenvalue in an eigenvalue problem respresents certain physical quantity. It is therefore necessary that the eigenvalues be real. It can be shown that the eigenvalues of an Hermitian matrix are real. Therefore, most physical quantities are represented by Hermitian matrices.

# Eigenvectors of Hermitian Matrix

3. Consider the three eigenvectors for the three-particle system,

$$x^1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad x^2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad x^3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

These eigenvectors are orthogonal, i.e.

$$\langle x^i | x^j \rangle = \delta_{ij}$$

In general, the eigenvectors of an Hermitian matrix corresponding to different eigenvalues are orthogonal.