Lecture 33

Topic

Transformation of Coordinates &
Change of Basis

Relevance

Linear transformation is common and important. Even though the physics laws are not changed by a change of basis, the mathematics involved of a given problem may be much simpler in certain basis.

Aims

To understand

1. invariance of physics laws under coordinate transformation;
2. the general properties of coordinate transformation or changing of basis;
3. orthogonal tranformation;
4. the application in eigenvalue problem.

Transformation of Coordinates: Examples

Cartesian $\Longleftrightarrow$ cylindrical (polar)

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Cartesian $\Longleftrightarrow$ spherical (polar)

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

Two body system

$$(x_1, y_1, z_1, x_2, y_2, z_2) \iff (X, Y, Z, x, y, z)$$

Here (X, Y, Z) is the center of mass position of the system and (x, y, z) are the relative coordinates.

Rotation

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}$$

Linear Transformations

Transformation between Cartesian coordiantes and cylindrical coordinates, or between Cartesian coordinates and spherical coordinates are non-linear. But between coordinates of individual particles and centre of mass and relative coordinates, and a rotation of cooridnate system are linear transformations.

A **linear transformation** is one in which each new variable is some linear combination of the old variables. In two dimensionals the linear transformation equations are

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

where a, b, c, and d are constants. Or

$$\vec{r}\,' = M\vec{r}$$

M tells us how to get the components of the vector $\vec{r} = \vec{r}\,'$ relative to axes x' and y' when we know its components relative to axes x and y.

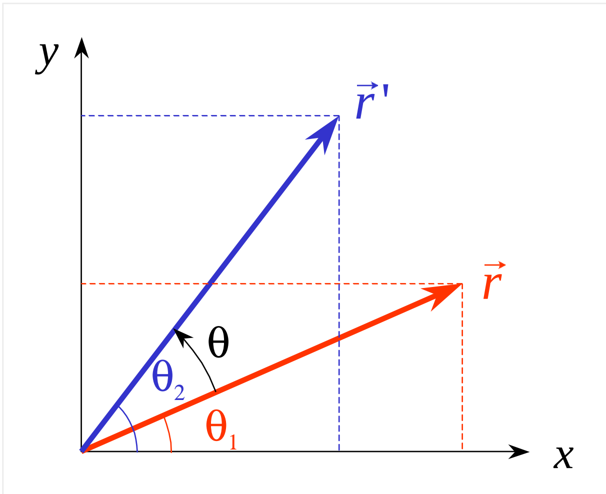
Rotation: Transform a Vector

A rotation (and other transformations) can be interpreted geometrically in two ways: the "change of cooridnate system (geometric language)" and "change of variables (algebraic language)".

Change of variables

$$x = r\cos\theta_1$$
$$y = r\sin\theta_1$$

$$x' = r\cos\theta_2$$
$$y' = r\sin\theta_2$$

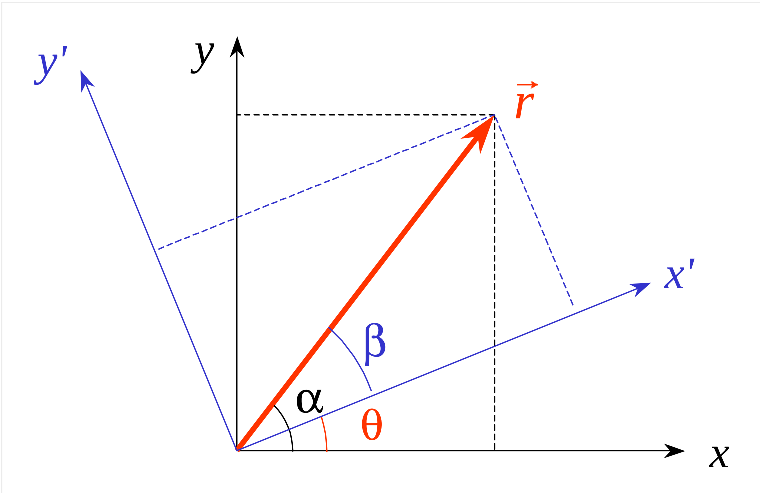


$$x' = r\cos(\theta_1+\theta) = r\cos\theta_1\cos\theta - r\sin\theta_1\sin\theta$$
$$y' = r\sin(\theta_1+\theta) = r\sin\theta_1\cos\theta + r\cos\theta_1\sin\theta$$

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}$$

Change of Coordinate System

Transformation of coordinates generally refer to change of coordinate system (change of basis).



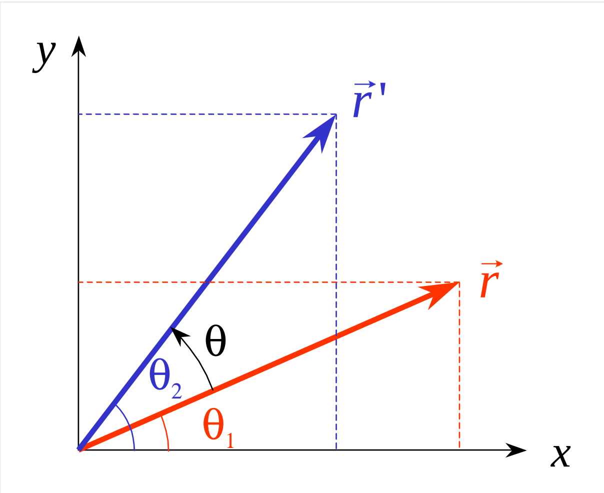
$$x = r\cos\alpha \qquad x' = r\cos\beta$$
$$y = r\sin\alpha \qquad y' = r\sin\beta$$

$$x' = r\cos(\alpha - \theta) = r\cos\alpha\cos\theta + r\sin\alpha\sin\theta$$

$$y' = r\sin(\alpha - \theta) = r\sin\alpha\cos\theta - r\cos\alpha\sin\theta$$

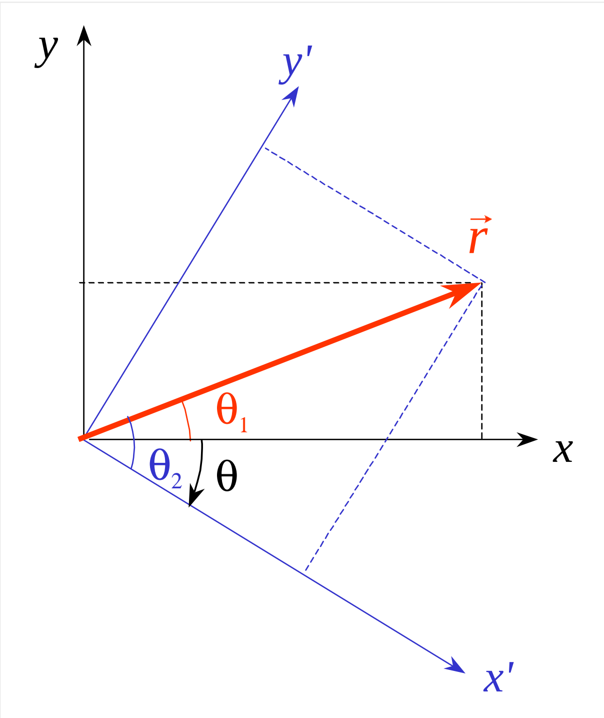
$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}$$

Change of Coordinates vs. Change of Variables



$$\vec{r}\,' = R(\theta)\vec{r}$$

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



$$\vec{r}\,' = R^{-1}(\theta)\vec{r}$$

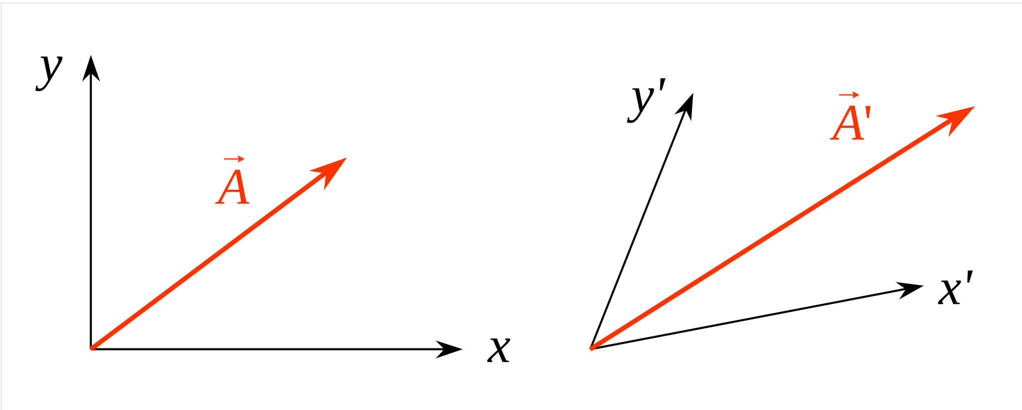
$$\vec{r}\,' = R^{-1}(-\theta)\vec{r}$$
$$\vec{r}\,' = R(\theta)\vec{r}$$

General Linear Transformation

In general, the axes x' and y' given by the linear transformation

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

are not perpendicular.



When they are, the equations represent a rotation and a, b, c, d can be written in terms of the rotation angle θ so that the equations become

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}$$

Such a linear transformation is called a *orthogonal transformation*.

Orthogonal Transformation

By definition, an orthogonal transformation is a linear transformation from x, y to x', y', such that

$$x^2 + y^2 = x'^2 + y'^2$$

That is, the length of a vector is not changed by an orthogonal transformation.

Since

$$\begin{array}{ccccc} x' & = & ax & + & by \\ y' & = & cx & + & dy \end{array}$$

$$\begin{aligned} x'^2 + y'^2 &= (ax + by)^2 + (cx + dy)^2 \\ &= (a^2 + c^2)x^2 + 2(ab + cd)xy + (b^2 + d^2)y^2 \end{aligned}$$

In order for $x'^2 + y'^2 = x^2 + y^2$, we must have

$$\begin{array}{c} a^2 + c^2 = 1 \\ b^2 + d^2 = 1 \\ ab + cd = 0 \end{array} \quad \Longrightarrow \quad \begin{array}{c} a = d = \cos\theta \\ b = -c = \sin\theta \end{array}$$

Orthogonal Matrix

The matrix M of an orthogonal transformation is called an *orthogonal matrix*. For an orthogonal matrix,

$$M^T M = I \quad \text{or} \quad M^T = M^{-1}$$

If

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$M^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$M^T M = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}$$

Therefore,

$$\begin{aligned} a^2 + c^2 &= 1 \\ b^2 + d^2 &= 1 \\ ab + cd &= 0 \end{aligned} \quad \Longleftrightarrow \quad M^T M = I$$

3D Example

Consider two basis sets:

1. $\hat{i}, \hat{j}, \hat{k}$ (Cartesian)
2. $\vec{a}, \vec{b}, \vec{c}$ (Basis vector of crystal lattice)

A vector $\vec{r}$ can be written in terms of either set

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$\vec{r} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$$

Assume that

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$
$$\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$$
$$\vec{c} = c_x\hat{i} + c_y\hat{j} + c_z\hat{k}$$

$$\begin{aligned}\vec{r} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} &= \alpha(a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \\ &+\beta(b_x\hat{i} + b_y\hat{j} + b_z\hat{k}) + \gamma(c_x\hat{i} + c_y\hat{j} + c_z\hat{k}) \\ &= (\alpha a_x + \beta b_x + \gamma c_x)\hat{i} + (\alpha a_y + \beta b_y + \gamma c_y)\hat{j} \\ &+(\alpha a_z + \beta b_z + \gamma c_z)\hat{k}\end{aligned}$$

$$\Longrightarrow \qquad \begin{aligned} x &= \alpha a_x + \beta b_x + \gamma c_x \\ y &= \alpha a_y + \beta b_y + \gamma c_y \\ z &= \alpha a_z + \beta b_z + \gamma c_z \end{aligned}$$

3D Example

Rewrite the equations in matrix form

$$\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} = \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

The first equation relates the basis (unit vectors) of two coordinate systems. The second equation relates the components of a vector in the two systems.

One matrix is the transpose of the other.

Change of Basis

Choose a basis $\hat{e}_i, i = 1, 2, \cdots, N$, a vector x in N dimensional linear space can be written as

$$x = x_1 \hat{e}_1 + x_2 \hat{e}_2 + \cdots + x_N \hat{e}_N = \sum_{i=1}^{N} x_i \hat{e}_i$$

Consider a change of basis

$$(\hat{e}_1, \hat{e}_2, \cdots, \hat{e}_N) \Rightarrow (\hat{e}'_1, \hat{e}'_2, \cdots, \hat{e}'_N)$$

For linear transformation,

$$\hat{e}'_j = \sum_{i=1}^{N} S_{ij} \hat{e}_i$$

where S_{ij} is the ith component of $\hat{e}'_j$ with respect to the old (unprimed) basis.

The physical facts described by the answer to a physical problem do not depend upon which basis we decide to use. For a vector x,

$$x = \sum_{i=1}^{N} x_i \hat{e}_i = \sum_{j=1}^{N} x'_j \hat{e}'_j$$

Change of Basis

$$\sum_{j=1}^{N} x'_j \hat{e}'_j = \sum_{j=1}^{N} x'_j \sum_{i=1}^{N} S_{ij} \hat{e}_i = \sum_{i=1}^{N} \left(\sum_{j=1}^{N} S_{ij} x'_j \right) \hat{e}_i$$

$$x_i = \sum_{j=1}^{N} S_{ij} x'_j$$

$$\boxed{x = Sx', \quad x' = S^{-1}x}$$

Consider the operator equation

$$\begin{aligned} y = Ax, & \qquad y' = A'x' \\ \Downarrow & \qquad (x = Sx', y = Sy') \\ Sy' = ASx' & \\ \Downarrow & \qquad \text{(multiply by } S^{-1} \text{ from left)} \\ y' = S^{-1}ASx' & \end{aligned}$$

$$\boxed{A' = S^{-1}AS}$$

This is referred as a similarity transformation.

Orthogonal Transformation

Consider

$$x' = Mx$$

$$\begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

The *norm* of x is given by

$$\begin{aligned} x^T x &= (x_1, x_2, \cdots, x_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= x_1^2 + x_2^2 + \cdots + x_n^2 \end{aligned}$$

$$x'^T x' = (Mx)^T (Mx) = x^T M^T M x$$

In order for $x'^T x' = x^T x$, there must be

$$M^T M = I \quad \text{or} \quad M^T = M^{-1}$$

The transformation matrix must be orthogonal!

Three-Particle Oscillation

Consider again the three-particle oscillation discussed in the last lecture,

$$\omega^1 = 0, \quad ,\omega^2 = \omega_0, \quad \omega^3 = \sqrt{3}\omega_0$$

$$x^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad x^2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad x^3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

The unnormalized eigenvectors will be used for simplicity.

Introducting the normal coordinates, u_1, u_2 and u_3.

$$\begin{aligned} u_1 &= x_1 + x_2 + x_3 \\ u_2 &= x_1 - x_3 \\ u_3 &= x_1 - 2x_2 + x_3 \end{aligned}$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$u = S^{-1}x$$

Three-Particle Oscillation

Consider

$$Ax = -\omega^2 x$$
$$u = S^{-1}x \quad \longrightarrow \quad x = Su$$
$$ASu = -\omega^2 Su$$

Multiply by S^{-1} from left

$$S^{-1}ASu = A'u = -\omega^2 u$$

$$S^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix} \rightarrow S = \frac{1}{6}\begin{pmatrix} 2 & 3 & 1 \\ 2 & 0 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

$$A = \omega_0^2 \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad A' = S^{-1}AS = -\omega_0^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$-\omega_0^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = -\omega^2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$0 = -\omega^2 u_1$$
$$-\omega_0^2 u_2 = -\omega^2 u_2$$
$$-3\omega_0^2 u_3 = -\omega^2 u_3$$

Decoupled!

Observation

Each column of S is an eigenvector of A. If normalized eigenvectors were used, then

$$S^{-1} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}$$

$$S = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ x^1 & x^2 & x^3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

In general,

$$Ax^j = \lambda_j x^j$$

$$S = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ x^1 & x^2 & \cdots & x^3 \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

$$A' = S^{-1}AS = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$$

Given a matrix A, if we construct the matrix S that has the eigenvectors of A as its columns, then the matrix $A' = S^{-1}AS$ is diagonal with the eigenvalues of A as the diagonal elements.