Fourier Series

Dirichlet conditions

The particular conditions that a function $f(x)$ must fulfil in order that it may be expanded as a Fourier series are known as the Dirichlet conditions, and may be summarized by the following points:

1. the function must be periodic;
2. it must be single-valued and continuous, except possibly at a finite number of finite discontinuities;
3. it must have only a finite number of maxima and minima within one periodic;
4. the integral over one period of $|f(x)|$ must converge.

If the above conditions are satisfied, then the Fourier series converge to $f(x)$ at all points where $f(x)$ is continuous.

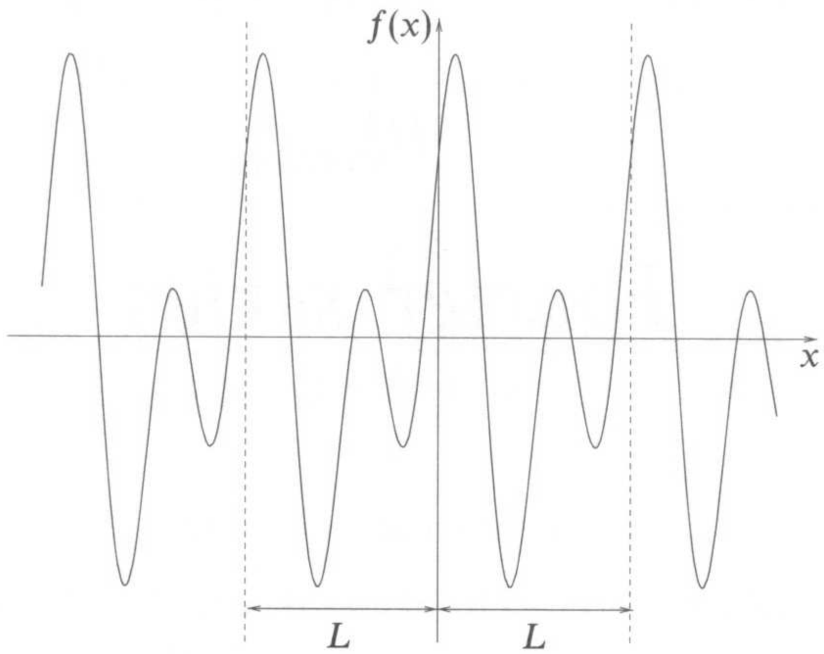

FIG. 1: An example of a function that may, without modification, be represented as a Fourier series.

Fourier coefficients

The Fourier series expansion of the function $f(x)$ is written as

$$f(x) = \frac{a}{2} + \sum_{r=1}^{\infty} \left[a_r \cos\left(\frac{2\pi rx}{L}\right) + b_r \sin\left(\frac{2\pi rx}{L}\right) \right] \tag{1}$$

where a_0, a_r and b_r are constants called the Fourier coefficients.

For a periodic function $f(x)$ of period L, the coefficients are given by

$$a_r = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi rx}{L}\right) dx \tag{2}$$

$$b_r = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \sin\left(\frac{2\pi rx}{L}\right) dx \tag{3}$$

where x_0 is arbitrary but is often taken as 0 or $-L/2$. The apparent arbitrary factor $1/2$ which appears in the a_0 term in Eq. (1) is included so that Eq. (2) may apply for $r = 0$ as well as $r > 0$.

The relations Eqs. (2) and (3) may be derived as follows.

Suppose the Fourier series expansion of $f(x)$ can be written as in Eq. (1),

$$f(x) = \frac{a}{2} + \sum_{r=1}^{\infty} \left[a_r \cos\left(\frac{2\pi r x}{L}\right) + b_r \sin\left(\frac{2\pi r x}{L}\right) \right]$$

Then multiplying by $\cos(2\pi p x/L)$, integrating over one full period in x and changing the order of summation and integration, we get

$$\begin{aligned} \int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi p x}{L}\right) dx &= \frac{a_0}{2} \int_{x_0}^{x_0+L} \cos\left(\frac{2\pi p x}{L}\right) dx \\ &+ \sum_{r=1}^{\infty} a_r \int_{x_0}^{x_0+L} \cos\left(\frac{2\pi r x}{L}\right) \cos\left(\frac{2\pi p x}{L}\right) dx \\ &+ \sum_{r=1}^{\infty} b_r \int_{x_0}^{x_0+L} \sin\left(\frac{2\pi r x}{L}\right) \cos\left(\frac{2\pi p x}{L}\right) dx \end{aligned} \tag{4}$$

Using the following orthogonality conditions,

$$\int_{x_0}^{x_0+L} \sin\left(\frac{2\pi r x}{L}\right)\cos\left(\frac{2\pi p x}{L}\right)dx = 0 \quad (5)$$

$$\int_{x_0}^{x_0+L} \cos\left(\frac{2\pi r x}{L}\right)\cos\left(\frac{2\pi p x}{L}\right)dx = \begin{cases} L, & r = p = 0 \\ \frac{1}{2}L, & r = p > 0 \\ 0, & r \neq p \end{cases} \quad (6)$$

$$\int_{x_0}^{x_0+L} \sin\left(\frac{2\pi r x}{L}\right)\sin\left(\frac{2\pi p x}{L}\right)dx = \begin{cases} 0, & r = p = 0 \\ \frac{1}{2}L, & r = p > 0 \\ 0, & r \neq p \end{cases} \quad (7)$$

we find that when $p = 0$, Eq. (4) becomes

$$\int_{x_0}^{x_0+L} f(x)\,dx = \frac{a_0}{2}L$$

When $p \neq 0$ the only non-vanishing term on the RHS of Eq. (4) occurs when $r = p$, and so

$$\int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi r x}{L}\right) dx = \frac{a_r}{2} L.$$

The other coefficients b_r may be found by repeating the above process but multiplying by $\sin(2\pi p x/L)$ instead of $\cos(2\pi p x/L)$.

Example

Express the square-wavefunction illustrated in the figure below as a Fourier series.

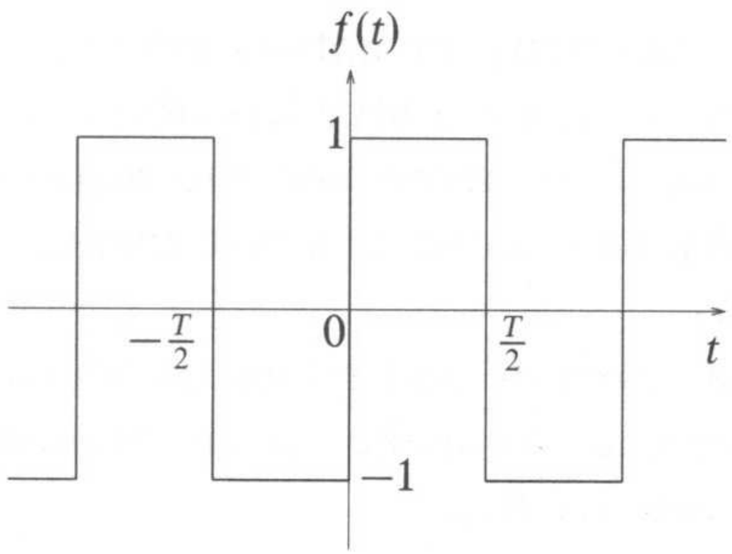

FIG. 2: A square-wavefunction.

The square wave may be represented by

$$f(t) = \begin{cases} -1 & \text{for } -\frac{1}{2}T \leq t < 0, \\ +1 & \text{for } 0 \leq t < \frac{1}{2}T. \end{cases}$$

Note that the function is an odd function and so the series will contain only sine terms. To evaluate the coefficients in the sine series, we use Eq. (3). Hence

$$\begin{aligned} b_r &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi rt}{T}\right)\,dt \\ &= \frac{4}{T}\int_{0}^{T/2} \sin\left(\frac{2\pi rt}{T}\right)\,dt \\ &= \frac{2}{\pi r}\left[1-(-1)^r\right]. \end{aligned}$$

Thus the sine coefficients are zero if r is even and equal to $4/\pi r$ if r is odd. hence the Fourier series for the square-wavefunction may be written as

$$f(t) = \frac{4}{\pi}\left(\sin\omega t + \frac{\sin 3\omega t}{3} + \frac{\sin 5\omega t}{5} + \cdots\right),$$

where $\omega = 2\pi/T$ is called the angular frequency.

Discontinuous functions

At a point of finite discontinuity, x_d, the Fourier series converges to

$$\frac{1}{2}\lim_{\epsilon\to 0}[f(x_d+\epsilon)+f(x_d-\epsilon)].$$

At a discontinuity, the Fourier series representation of the function will overshoot its value. Although as more terms are included the overshoot moves in position arbitrarily close to the discontinuity, it never disappears even in the limit of an infinite number of terms. This behavior is known as Gibbs' phenomenon.

Example

Find the value to which the Fourier series of the square-wavefunction converges at $t = 0$.

Answer

The function is discontinuous at $t = 0$, and we expect the series to converge to a value half-way between the upper and lower values; zero in this case. Considering the Fourier series of this function, we see that all the terms are zero and hence the Fourier series converges to zero as expected.

The Gibbs phenomenon is shown below.

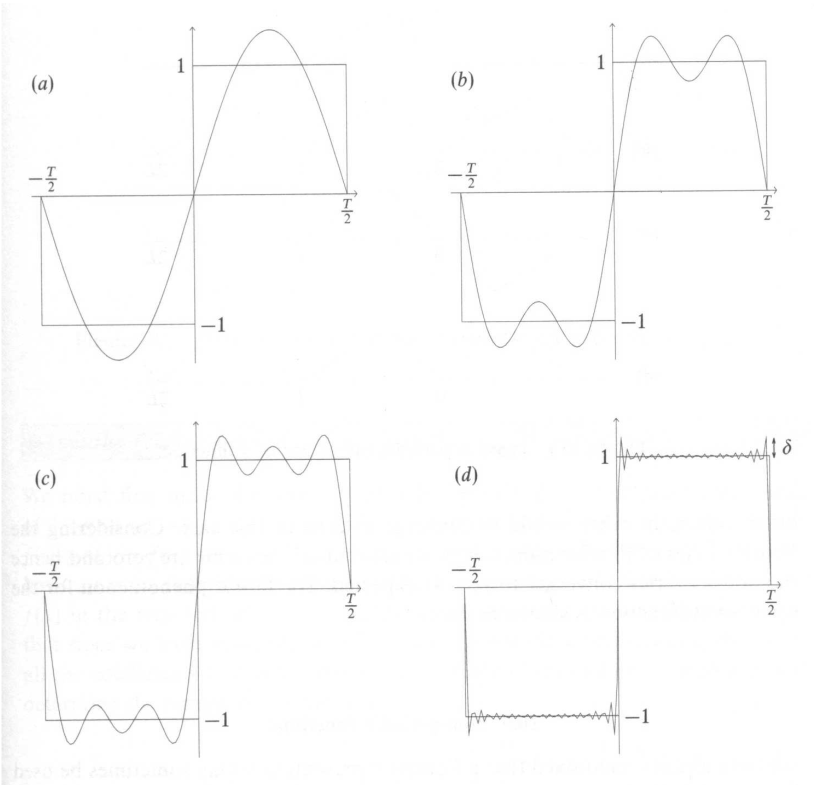

FIG. 3: The convergence of a Fourier series expansion of a square-wavefunction, including (a) one term, (b) two terms, (c) three terms and (d) 20 terms. The overshoot δ is shown in (d).

Non-periodic functions

Figure 4(b) shows the simplest extension to the function shown in Figure 4(a). However, this extension has no particular symmetry. Figures 4(c), (d) show extensions as odd and even functions respectively with the benefit that only sine or cosine terms appear in the resulting Fourier series.

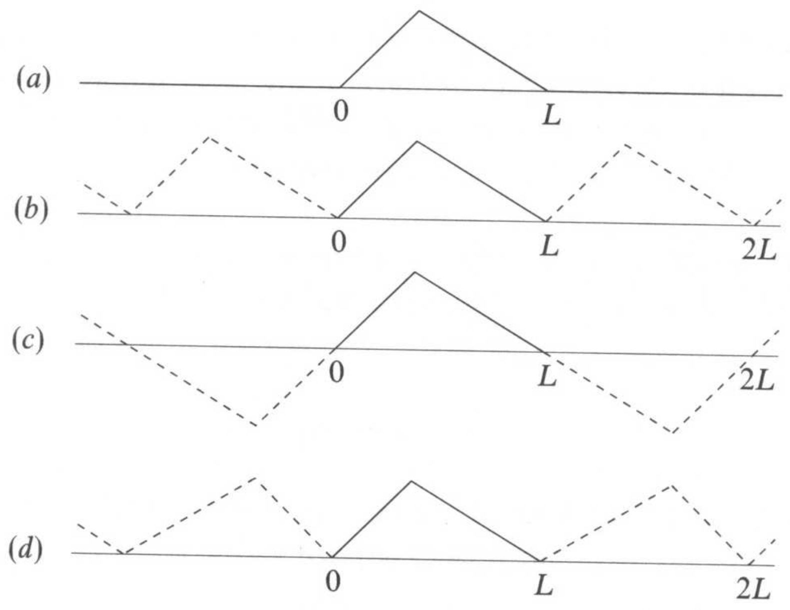

FIG. 4: Possible periodic extensions of a function.

Example

Find the Fourier series of $f(x) = x^2$ for $0 < x \leq 2$.

Answer

We must first make the function periodic. We do this by extending the range of interest to $-2 < x \leq 2$ in such a way that $f(x) = f(-x)$ and then letting $f(x + 4k) = f(x)$ where k is any integer.

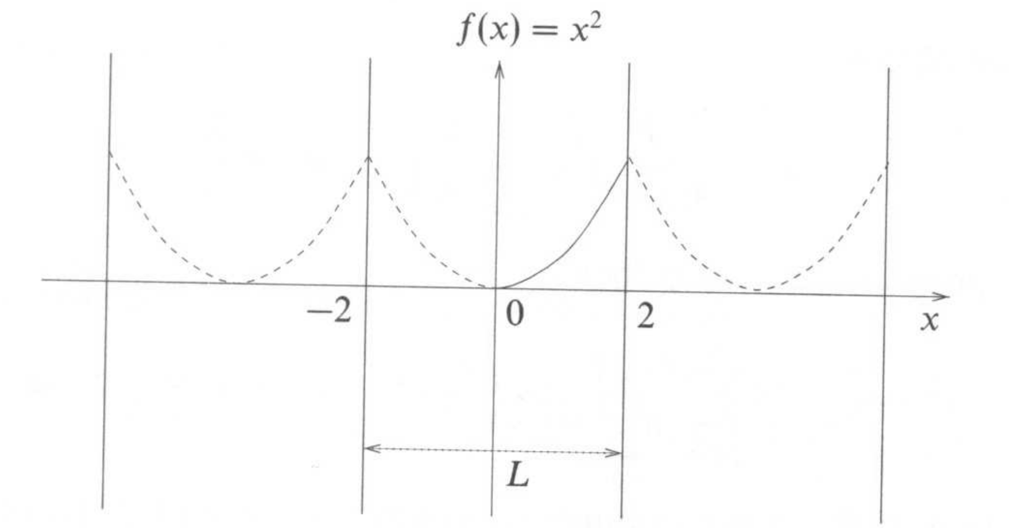

FIG. 5: $f(x) = x^2$, $0 < x \leq 2$, with extended range and periodicity.

Now we have an even function of period 4. Thus, all the coefficients b_r will be zero. Now we apply Eqs. (2) and (3) with $L = 4$ to determine the remaining coefficients:

$$a_r = \frac{2}{4}\int_{-2}^{2} x^2 \cos\left(\frac{2\pi r x}{4}\right) dx = \frac{4}{4}\int_{0}^{2} x^2 \cos\left(\frac{\pi r x}{2}\right) dx,$$

Thus,

$$\begin{aligned}
a_r &= \left[\frac{2}{\pi r} x^2 \sin\left(\frac{\pi r x}{2}\right)\right]_0^2 - \frac{4}{\pi r}\int_0^2 x \sin\left(\frac{\pi r x}{2}\right) dx \\
&= \frac{8}{\pi^2 r^2}\left[x \cos\left(\frac{\pi r x}{2}\right)\right]_0^2 - \frac{8}{\pi^2 r^2}\int_0^2 \cos\left(\frac{\pi r x}{2}\right) dx \\
&= \frac{16}{\pi^2 r^2}\cos \pi r \\
&= \frac{16}{\pi^2 r^2}(-1)^r.
\end{aligned}$$

Since the expression for a_r has r^2 in its denominator, to evaluate a_0 we must return to the original definition,

$$a_r = \frac{2}{4}\int_{-2}^{2} f(x)\cos\left(\frac{\pi r x}{2}\right)\, dx.$$

From this we obtain

$$a_0 = \frac{2}{4}\int_{-2}^{2} x^2\, dx = \frac{4}{4}\int_{0}^{2} x^2\, dx = \frac{8}{3}.$$

The final expression for $f(x)$ is then

$$x^2 = \frac{4}{3} + 16\sum_{r=1}^{\infty}\frac{(-1)^r}{\pi^2 r^2}\cos\left(\frac{\pi r x}{2}\right), \quad \text{for } 0 < x \leq 2. \tag{8}$$

Integration and differentiation

Example

Find the Fourier series of $f(x) = x^3$ for $0 < x \leq 2$.

Answer

If we integrate Eq. (8) term by term, we obtain

$$\frac{x^3}{3} = \frac{4}{3}x + 32 \sum_{r=1}^{\infty} \frac{(-1)^r}{\pi^3 r^3} \sin\left(\frac{\pi r x}{2}\right) + c,$$

where c is an arbitrary constant. We have not yet found the Fourier series for x^3 because the term $\frac{4}{3}x$ appears in the expansion. However, now differentiating our expression for x^2, we obtain

$$2x = -8 \sum_{r=1}^{\infty} \frac{(-1)^r}{\pi r} \sin\left(\frac{\pi r x}{2}\right).$$

We can now write the full Fourier expansion of x^3 as

$$
\begin{aligned}
x^3 \quad &= \quad -16 \sum_{r=1}^{\infty} \frac{(-1)^r}{\pi r} \sin\left(\frac{\pi r x}{2}\right) \\
&\quad\; +96 \sum_{r=1}^{\infty} \frac{(-1)^r}{\pi^3 r^3} \sin\left(\frac{\pi r x}{2}\right) + c
\end{aligned}
$$

We can find the constant c by considering $f(0)$. At $x = 0$, our Fourier expansion gives $x^3 = c$ since all sine terms are zero, and hence $c = 0$.

Complex Fourier series

Using $\exp(irx) = \cos rx + i \sin rx$, the complex Fourier series expansion is written as

$$f(x) = \sum_{r=-\infty}^{\infty} c_r \exp\left(\frac{2\pi irx}{L}\right), \tag{9}$$

where the Fourier coefficients are given by

$$c_r = \frac{1}{L} \int_{x_0}^{x_0+L} f(x) \exp\left(-\frac{2\pi irx}{L}\right) dx \tag{10}$$

This relation can be derived by multiplying Eq. (9) by $\exp(-2\pi ipx/L)$ before integrating and using the orthogonality relation

$$\int_{x_0}^{x_0+L} \exp\left(-\frac{2\pi ipx}{L}\right) \exp\left(\frac{2\pi irx}{L}\right) dx = \begin{cases} L, & r = p \\ 0, & r \neq p \end{cases}$$

The complex Fourier coefficients have the following relations with the real Fourier coefficients

$$\begin{aligned} c_r &= \frac{1}{2}(a_r - ib_r), \qquad (11) \\ c_{-r} &= \frac{1}{2}(a_r + ib_r). \end{aligned}$$

Example

Find a complex Fourier series for $f(x) = x$ in the range $-2 < x < 2$.

Answer

Using Eq. (10),

$$\begin{aligned}
c_r &= \frac{1}{4}\int_{-2}^{2} x \exp\left(-\frac{\pi i r x}{2}\right) dx \\
&= \left[-\frac{x}{2\pi i r} \exp\left(-\frac{\pi i r x}{2}\right)\right]_{-2}^{2} \\
&\quad + \int_{-2}^{2} \frac{1}{2\pi i r} \exp\left(-\frac{\pi i r x}{2}\right) dx \\
&= -\frac{1}{\pi i r}[\exp(-\pi i r) + \exp(\pi i r)] \\
&\quad + \left[\frac{1}{r^2\pi^2} \exp\left(-\frac{\pi i r x}{2}\right)\right]_{-2}^{2} \\
&= \frac{2i}{\pi r}\cos \pi r + \frac{2i}{r^2\pi^2}\sin \pi r \\
&= \frac{2i}{\pi r}(-1)^r.
\end{aligned}$$

Hence

$$x = \sum_{r=-\infty}^{\infty} \frac{2i(-1)^r}{r\pi} \exp\left(\frac{\pi i r x}{2}\right).$$

Parseval's theorem

Parseval's theorem gives a useful way of relating the Fourier coefficients to the function that they describe. Essentially a conservation law, it states that

$$\begin{aligned} \frac{1}{L}\int_{x_0}^{x_0+L} |f(x)|^2\,dx &= \sum_{r=-\infty}^{\infty} |c_r|^2 \\ &= \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_{r=1}^{\infty}(a_r^2 + b_r^2). \end{aligned} \qquad (12)$$

This says that the sum of the moduli squared of the complex Fourier coefficients is equal to the average value of $|f(x)|^2$ over one period.

Proof of Parseval's theorem

Let us consider two functions $f(x)$ and $g(x)$, which are (or can be made) periodic with period L, and which have Fourier series (expressed in complex form)

$$\begin{aligned} f(x) &= \sum_{r=-\infty}^{\infty} c_r \exp\left(\frac{2\pi irx}{L}\right), \\ g(x) &= \sum_{r=-\infty}^{\infty} \gamma_r \exp\left(\frac{2\pi irx}{L}\right) \end{aligned}$$

where c_r and γ_r are the complex Fourier coefficients of $f(x)$ and $g(x)$ respectively. Thus,

$$f(x) * g(x) = \sum_{r=-\infty}^{\infty} c_r g^*(x) \exp\left(\frac{2\pi irx}{L}\right)$$

Integrating this equation with respect to x over the interval $(x_0, x_0 + L)$, and dividing by L, we find

$$\frac{1}{L}\int_{x_0}^{x_0+L} f(x)g^*(x)dx =$$

$$\sum_{r=-\infty}^{\infty} c_r \frac{1}{L}\int_{x_0}^{x_0+L} g^*(x)\exp\left(\frac{2\pi i r x}{L}\right)\,dx$$

$$= \sum_{r=-\infty}^{\infty} c_r \left[\frac{1}{L}\int_{x_0}^{x_0+L} g(x)\exp\left(\frac{-2\pi i r x}{L}\right)\,dx\right]^*$$

$$= \sum_{r=-\infty}^{\infty} c_r \gamma_r^*.$$

Finally, if we let $g(x) = f(x)$, we obtain Parseval's theorem (Eq. (12)).

Example

Using Parseval's theorem and the Fourier series for $f(x) = x^2$, calculate the sum $\sum_{r=1}^{\infty} r^{-4}$.

Answer

Firstly, we find the average value of $|f(x)|^2$ over the interval $-2 < x \leq 2$,

$$\frac{1}{4}\int_{-2}^{2} x^4 \, dx = \frac{16}{5}.$$

Now we evaluate the RHS of Eq. (12):

$$\left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_{1}^{\infty} a_r^2 + \frac{1}{2}\sum_{1}^{\infty} b_n^2 = \left(\frac{4}{3}\right)^2 + \frac{1}{2}\sum_{r=1}^{\infty}\frac{16^2}{\pi^4 r^4}.$$

Equating the two expression, we find

$$\sum_{r=1}^{\infty}\frac{1}{r^4} = \frac{\pi^4}{90}.$$