Lecture 1: RNS Formulation of Superstring Theory

When a particle traverses spacetime, it sweeps out a worldline. A similar traversal by a string generates naturally a worldsheet. Let $X_\mu(\sigma,\tau)$ be the position vector of a string in $D-1$ space-like dimensions. The simplest bosonic action can then be written in conformal gauge as

$$S_B = \frac{1}{2\pi}\int d\sigma d\tau \partial_\alpha X^\mu(\sigma,\tau)\partial^\alpha X_\mu(\sigma,\tau). \tag{1}$$

Indeed we expect the string coordinates $X^\mu(\sigma,\tau)$ to be invariant under a reprametrization and it turns out that this can be realized by a super Virasoro algebra. The action (1) acts in D-minskowski spacetime. Consistency of special relativity and quantum mechanics requires $D = 26$. This consistency requirement arises from the need to ensure that the Lorentz generators constructed in the theory can close properly.

Bosonic string has problems

- The particle spectrum contains a tachyon with $m^2 < 0$ but whose norm is positive definite.
- It is inconsistent at one loop calculation due to the tachyon.
- there are no spacetime fermions and so it cannot be used to describe Nature realistically.

There are two ways of introducing superstrings:

- Green-Schwarz string. This string has genuine ten-dimensional spinor fields but the theory is noncovariant.
- Ramond-Neveu-schwarz (RNS) string. This formulation is manifestly covariant but the ten-dimensional spacetime supersymmetry is not obvious.

We next introduce a fermionic field ψ^μ to the bosonic string action. This fermionic field ψ^μ is essentially a worldsheet spinor and can be expressed as a doublet

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix} \tag{2}$$

but acts as a vector under $SO(1, D-1)$ Lorentz group. The action is now written as

$$S = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau \int_0^\pi d\sigma \{\partial_\alpha X^\mu(\sigma,\tau)\partial^\alpha X_\mu(\sigma,\tau) - i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu\} \tag{3}$$

where $\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $\bar{\psi}^\mu = \psi^T \rho^0$ with $\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$ in the Majorana repesentation.

This action possesses a supersymmetric invariance

$$\begin{aligned} \delta X^\mu &= \epsilon X^\mu & \text{(4a)} \\ \delta \psi^\mu &= -i\rho^\alpha \partial_\alpha X^\mu \epsilon. & \text{(4b)} \end{aligned}$$

We next introduce worldsheet light cone coordinates, $\partial_\pm = \partial_\tau \pm \partial_\sigma$. In this formalism, the fermionic part of the action reads

$$S_F = \frac{1}{\pi} \int d\tau \int d\sigma \{\psi^\mu_- \partial_+ \psi_{-\mu} + \psi^\mu_+ \partial_- \psi_{+\mu}\} \tag{5}$$

so that

$$\delta S_F = \frac{1}{\pi} \int d\tau \int d\sigma \{\delta\psi^\mu_- \partial_+ \psi_{-\mu} + \delta\psi^\mu_+ \partial_- \psi_{+\mu}\} + \frac{1}{2\pi} \int d\tau [\psi_- \delta\psi_- - \psi_+ \delta\psi_+]^\pi_0. \tag{6}$$

Invariance of the fermionic action requires $\partial_+ \psi_- = 0 = \partial_- \psi_+$ and the imposition of the boundary conditions at $\sigma = 0$ and $\sigma = \pi$ to kill the surface terms. We can always choose $\psi_+(\pi, \tau) = \psi_-(\pi, \tau)$. However, at $\sigma = 0$, we have the following two distinct boundary conditions:

1. $\psi_+(0, \tau) = \psi_-(0, \tau)$, Ramond sector;
2. $\psi_+(0, \tau) = -\psi_-(0, \tau)$, Neveu-Schwarz sector.

For closed strings, the boundary conditions are given by

1. $\psi_\pm(0, \tau) = \psi_\pm(2\pi, \tau)$, Ramond sector;
2. $\psi_\pm(0, \tau) = -\psi_\pm(2\pi, \tau)$, Neveu-Schwarz sector.

For the spectrum of states, we recall that superstrings are invariant under super-diffeomorphisms. We need to impose constraints on our state space and these constraints are provided by the super Virasoro algebra. The Fourier expansion of the fermion strings in the Ramond and Neveu-Schwarz sectors are given by

$$\begin{aligned} \psi^\mu_- &= \sum_{n \in \mathbf{Z}} d^\mu_n \exp(-2in(\tau - \sigma)) & \text{(7a)} \\ \psi^\mu_+ &= \sum_{n \in \mathbf{Z}} \tilde{d}^\mu_n \exp(-2in(\tau + \sigma)) & \text{(7b)} \end{aligned}$$

and

$$\begin{aligned} \psi^\mu_- &= \sum_{r \in \mathbf{Z} + \frac{1}{2}} b^\mu_r \exp(-2ir(\tau - \sigma) & \text{(8a)} \\ \psi^\mu_+ &= \sum_{r \in \mathbf{Z} + \frac{1}{2}} \tilde{b}^\mu_r \exp(-2ir(\tau + \sigma) & \text{(8b)} \end{aligned}$$

respectively with the commutation relations

$$\begin{aligned} \{d_n^\mu, d_m^\nu\} &= \eta^{\mu\nu}\delta_{n+m}, \text{etc} && (9a)\\ \{b_r^\mu, b_s^\nu\} &= \eta^{\mu\nu}\delta_{r+s}, \text{etc} \ . && (9b) \end{aligned}$$

Let $\mathcal{F}^{\mathcal{B}}$ and $\tilde{\mathcal{F}}^{B}$ be the left and right movers for the bosons. Then for open strings, one sees that the state space is given by

$$\mathcal{F}_{\text{open}} = \mathcal{F}^B \otimes (\mathcal{F}^R \oplus \mathcal{F}^{NS}). \tag{10}$$

In the case of closed strings, the space is given by

$$\mathcal{F}_{\text{closed}} = \mathcal{F}_{\text{open}} \otimes \tilde{\mathcal{F}}_{\text{open}}. \tag{11}$$

With this formalism, one can get rid of negative norm states and creates states of higher masses. For instance in the Neveu-Schwarz sector, one demand that

$$b_r|0, NS >= 0 \tag{12}$$

for $r = \frac{1}{2}, \frac{3}{2}, \cdots$ and creates higer mass states through $b_{-r_1}b_{-r_2}\cdots b_{-r_N}|0, NS >$. It turns out that all excited states in the Neveu Schwarz sector are spacetime bosons. A similar analysis for the Ramond sector shows that the vacua transform as a spinor of spin$(1, D-1)$.

The energy-momentum tensor in the theory forms a closed algebra. From the Fourier modes of the energy-momentum tensor, one can construct the $N = 1$ super Virasoro generators (L_m, G_r) of the left moving sector for instance, satisfying

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{8}(m^3 - m)\delta_{m+n,0} && (13a)\\ [L_m, G_r] &= (\frac{m}{2} - r)G_{m+r} && (13b)\\ \{G_r, G_s\} &= 2G_{r+s} + \frac{c}{2}(r^2 - \epsilon_A)\delta_{r+s,0} && (13c) \end{aligned}$$

where both L_n, G_r belong to either the R or NS sector and

$$\begin{aligned} \epsilon_R &= 0 && (14a)\\ \epsilon_{NS} &= \frac{1}{4}. && (14b) \end{aligned}$$

There is a similar algebra $(\tilde{L}_m, \tilde{G}_r)$ for the right moving sector