Lecture 2: Spectrum of Physical States and GSO Projection

In the previous lecture, we have discussed how to eliminate and avoid negative norm states through the application of the super-Virasoro algebra by combining the NS and R sector. In this lecture, we shall first consider the spectrum of the physical states at low mass and show the existence of a state with mass^2 < 0, call a tachyon. To remove this unphysical state, we shall consider a technique proposed by Gliozzi, Scherk and Olive in the seventies, commonly known as the GSO projection.

Consider the Ramond sector for the open string. We shall consider left movers. The ground state $|0, \alpha, R>$ satisfies the relation, for $n > 0$,

$$\alpha_n^\mu |0, \alpha, R> = 0 = d_n^\mu |0, \alpha, R> . \tag{1}$$

From the commutation relations of the operators G_r and L_n in the previous lecture, you can easily show that $G_0^2 = L_0$. Since the ground state satisfies $G_0|0, \alpha, R> = 0$, it follows that $L_0|0, \alpha, R> = 0$. Further,

$$\begin{aligned} 0 &= G_0|0, k, \alpha, R>, &(2a)\\ &= \alpha_0^\mu d_{0\mu}|0, k, \alpha, R>, &(2b)\\ &= k^\mu \Gamma_\mu |0, k, \alpha, R>, &(2c) \end{aligned}$$

giving us the massless Dirac equation. [1] To obtain further excited massive states, one needs to consider the anti-symmetric polarization tensor.

In the Neveu-Schwarz sector, the ground state obeys

$$\begin{aligned} b^\mu_{r+1/2}|0, k, NS> &= 0, \qquad r > 0, &(3a)\\ G_r|0, k, NS> &= 0 \qquad r \in \mathrm{Z} + \frac{1}{2}. &(3b) \end{aligned}$$

The physical state is subject to the mass shell condition

$$\begin{aligned} (L_0 - \frac{1}{2})|0, k, NS> &= 0 &(4a)\\ \Rightarrow (\frac{1}{2}\alpha_0^2 - \frac{1}{2})|0, k, NS> &= 0 &(4b)\\ \Rightarrow (\frac{1}{2}k^2 - \frac{1}{2})|0, k, NS> &= 0 &(4c)\\ \Rightarrow k^2 - 1 = -m^2 - 1 &= 0, &(4d) \end{aligned}$$

so that this state has (mass^2 < 0) and is indeed a tachyon.

The first excited state $|\zeta, k, NS>$ is obtained through the relation

$$|\zeta, k, NS> = \zeta_\mu b^\mu_{-1/2}|0, k, NS> \tag{5}$$

[1] Historically, Ramond considered the whole idea in reverse; begining with the massless Dirac equation and applying it to the superstring theory.

and satisfies

$$
\begin{aligned}
0 = (L_0 - \frac{1}{2})|\zeta, k, NS > &= (L_0 - \frac{1}{2})\zeta_\mu b^\mu_{-1/2}|0, k, NS > \\
\Rightarrow k^2 &= 0, && \text{(6a)} \\
0 &= L_1|\zeta, k, NS > \\
\Rightarrow \zeta \cdot k &= 0 && \text{(6b)}
\end{aligned}
$$

Eq(6a) and Eq(6b) give rise to a massless vector state and a transversality condition. All other physical states can be obtained through the the direct application of the operators a^i_{-r}, b^i_{-r} and d^i_{-r} where $i = 1, \cdots, 8$.

For closed strings, we need to consider left and right movers in order to construct the complete spectrum:

$$p_L = p_R; \qquad \alpha^\mu_0 = \tilde{\alpha}^\mu_0 \tag{7}$$

The construction is easily done. Here, we shall summarize the result.

- NS-NS bosons
  - Ground state is a tachyon.
  - The massless states are graviton, $g_{\mu\nu}$, antisymmteric tensor, $b_{\mu\nu}$ and the dilaton ϕ.
- NS-R (or R-NS) fermions
  - No tachyon, since the Ramond sector always allow mass$^2 \geq 0$.
  - The massless ground state is reducible to the gravitino, and the dilatono.
- R-R bosons
  - No tachyon.
  - The massless states $|0, k, a, R >_L \otimes |0, k, b, R >_R$ decompose as a sum of the tensor representation of the group $SO(1, 9)$.

By combining the NS and the R sectors, we have constrcuted a string theory with bosons (e.g. the massless graviton) and fermions (e.g. gravitino). Despite the removal of negative norm states through the mass shell condition of the super Virasoro algebra, we still have states like tachyons. As it stands, this tachyon has no fermionic partner and the theory is essentially not space-time supersymmetric. To produce a spectrum with space-time supersymmetry, we need to consider the GSO projection.

We define the GSO projection as the worldsheet fermion operator F in the form of $(-1)^F$. For the NS sector, we may represent $(-1)^F$, for the left movers, by

$$(-1)^F = (-1)^{h-1}, \tag{8}$$

where $h = \sum_{r \in 1/2 + Z^+} b_{-r} \cdot b_r$. For the Ramond sector,

$$(-1)^F = \Gamma_{11} (-1)^{h_r} \tag{9}$$

where $h_r = \sum_{n>0} d_{-n} \cdot d_n$. There is a similar representation for right movers. To obtain space-time supersymmetry, we recall that in the NS sector, we have 8 physical degrees of freedom for A_μ and since in general a spinor in 10 dimensions has $2^{D/2} = 32$ complex components or 64 real components, we need to impose both the Majorana and Weyl conditions. This reduces the dimensionality to 16 but the imposition of Dirac equation eliminates half the modes giving 8 physical modes. These 8 fermionic states are the superpartners of the 8 bosonic degrees of freedom carried by A_μ.

For the Ramond sector, we note that in the Clifford algebra defined by $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$, one finds that the chirality matrix Γ_{11} defined by

$$\Gamma_{11} \equiv \Gamma^0 \Gamma^1 \cdots \Gamma^9 \tag{10}$$

anti-commutes with all the generators in the Dirac spinor representation. For $D = 2 \pmod 8$, a spinor can be reduced to a Weyl-Majorana spinor. This Weyl-Majorana condition effectively reduces the dimensionality of the ground state in the Ramond sector to an 8-component spinor. Having analyzed the GSO projection in a single sector of the RNS string theory, we can next consider closed oriented strings. Although a naive listing gives four possibilities, a more detailed analysis using parity operator shows that there are essentially two inequivalent closed string theories defined by the Ramond ground states:

- Type IIA, a non-chiral string theory in which the left-right Ramond vacua have opposite chirality and the theory is parity invariant under the exchange of left-right movers.
- Type IIB, a chiral string theory in which both the left-right Ramond vacua have the same chirality.