Lecture 3: Coupling of RR Fields to D-Branes

We recall that the imposition of Majorana-Weyl condition on the Ramond vacua yields

$$F_{\alpha,\beta}|0,k,\alpha,R>_L\ |0,k,\beta,R>_R \tag{1}$$

where $\alpha,\beta = 1,2,\cdots 16$. The bispinor $F_{\alpha,\beta}$ can be decomposed in a complete basis of all antisymmetric gamma-matrix products as

$$F_{\alpha,\beta} = \delta_{\alpha,\beta} + \sum_{k=1}^{10} \frac{i^k}{k!} F_{\mu_1\cdots\mu_k}(\Gamma^{\mu_1\cdots\mu_k})_{\alpha,\beta} \tag{2}$$

where $\Gamma^{\mu_1\cdots\mu_k} = \Gamma^{[\mu_1}\cdots\Gamma^{\mu_k]}$ and $F_{\mu_1\cdots\mu_k}$ is an antisymmetric Lorentz tensor. We also recall that the RR-vacua has definite chirality and this implies:

$$\Gamma_{11}F = -F\Gamma_{11} = F \qquad \text{Type IIA} \tag{3a}$$

$$\Gamma_{11}F = +F\Gamma_{11} = F \qquad \text{Type IIB.} \tag{3b}$$

Hence the antisymmetric tensors $F_{\mu_1\cdots\mu_k}$ are not independent. To write the constraints on $F_{\mu_1\cdots\mu_k}$, we note

$$\Gamma_{11}\Gamma^{\mu_1\cdots\mu_k} = \frac{(-)^{[k/2]}}{(10-k)!}\epsilon^{\mu_1\cdots\mu_{10}}\Gamma_{\mu_{k+1}\cdots\mu_{10}} \tag{4a}$$

$$\Gamma^{\mu_1\cdots\mu_k}\Gamma_{11} = \frac{(-)^{[(k+1)/2]}}{(10-k)!}\epsilon^{\mu_1\cdots\mu_{10}}\Gamma_{\mu_{k+1}\cdots\mu_{10}}. \tag{4b}$$

For type IIA, to satisfy eq(3a), we need k to be even, and similarly for type IIB, to satisfy eq(3b), k has to be odd. The antisymmetric field tensors satisfy

$$F\Gamma_{11} = -F \qquad \text{Type IIA} \tag{5}$$

which yields $F^{\mu_1\cdots\mu_k} = \frac{(-)^{[(k+1)/2]}}{(10-k)!}\epsilon^{\mu_1\cdots\mu_{10}}F_{\mu_{k+1}\cdots\mu_{10}}$ or $F_k = -*F_{(10-k)}$ with k even. For type IIB, the analogous equations are

$$F\Gamma_{11} = +F \tag{6a}$$

$$\text{or,}\ F_k = +*F_{(10-k)} \tag{6b}$$

with k odd.

We next recall that $F_{\alpha,\beta}$ has 16×16 components. We check that it has the correct components for type II A and type II B strings.

String Type	Independent Tensors	Number of components
Type II A	F_0 F_2 F_4	1 $\frac{10\times 9}{2!}$ $\frac{10\times 9\times 8\times 7}{4!}$ Total: 256
Type II B	F_1 F_3 $F_5 = *F_5$ (self dual)	10 $\frac{10\times 9\times 8}{3!}$ $\frac{10\times 9\times 8\times 7\times 6}{2\times 5!}$ Total: 256

Field Equations

The super Virasoro condition $G_0|\text{phys}>= 0$ yields

$$k_\mu \Gamma^\mu F = F k_\mu \Gamma^\mu = 0 \tag{7}$$

which ultimately leads to two equations

$$dF = 0 = d * F. \tag{8}$$

We note that for RR fields, the free massless equation $dF = 0$ and the Bianchi identity $d * F = 0$ appear on the same footing showing explicit duality.

From Bianchi identity,

$$F_{\mu_1\cdots\mu_k} = \frac{1}{(k-1)!}\partial_{[\mu_1} A_{\mu_2\cdots\mu_k]}$$

where $A_{\mu_1\cdots\mu_k}$ is the completely antisymmetric $U(1)$ field so that $F_{(k)} = dA_{(k-1)}$. Hence we have the following RR-fields

String Type	RR Fields	Number of components
Type II A	$f_{(0)}$ $A^{(1)}_\mu$ $A^{(3)}_{\mu\nu\lambda}$	0 8 $\frac{8\times7\times6}{3!}$ Total: 64
Type II B	$A^{(0)}$ $A^{(2)}_{\mu\nu}$ $A^{(4)}_{\mu\nu\lambda\delta}$ self-dual	1 $\frac{8\times7}{2!}$ $\frac{8\times7\times6\times5}{2\times4!}$ Total: 64

With this, we can list the D-brane couplings of type-II strings. Note that in general for tensor field $A^{(p+1)}$ with electric coupling to D-p brane, we also have the coupling of the dual $\tilde{A}^{q+1}$ to D-q brane via magnetic couplings. Indeed, we have

$$A^{(p+1)} \to dA_{(p+2)} \to d\tilde{A}_{(8-p)} \to \tilde{A}^{(7-p)} \tag{9}$$

so that the tensor field $A^{(p+1)}$ couples to D-p brane while its dual $\tilde{A}^{7-p}$ couples to a $D-(6-p)$ brane. We have the following D-brane couplings:

String Type	Couplings
Type II A	$F_{(0)} \to \tilde{F}_{(10)} \to \tilde{A}^{(9)}$ couples to 8-brane
	$A^{(1)}_{\mu}$ couples to 0-brane $\tilde{A}^{(7)}$ couples to 6-brane
	$A^{(3)}$ couples to 2-brane $\tilde{A}^{(5)}$ couples to 4-brane
Type II B	$A^{(0)}$ couples to -1-brane $\tilde{A}^{(8)}$ couples to 7-brane
	$A^{(2)}$ couples to 1-brane $\tilde{A}^{(6)}$ couples to 5-brane
	$A^{(4)}$ couples to 3-brane (self-dual)

Finally, we summarize everything in a 'brane-scan':

Type II A		0_D		2_D		4_D		6_D		8_D
Type II B	-1_D		1_D		3^+_D		5_D		7_D	
Type I			1_D				5_D			

Massive Spectra of Type II A and II B

Type II A and Type II B strings differ only for the massless states; the massive states are identical. While the massless states can be chiral, the massive spinor states cannot be chiral. For example, at the first excited level in the Ramond sector, we have states of opposite chirality, namely

$$\alpha^i_{-1}|0,k,\alpha,R>, \qquad d^i_{-1}|0,k,\alpha,R>$$

which combine into a maasive representation of a Majorana fermion. This means that the excited massive states are insensitive to the choice of the massless ground states.

If we compactify one space dimension with radius R_A for type II A and with radius R_B for type II B, then we can show that the theories are identical, including the massless sector, provided

$$R_A R_B = \alpha'. \tag{10}$$

This symmetry is called T-duality.

Coupling to D-brane

All the tensor gauge fields $A^{(k)}$ are $U(1)$ gauge fields and hence their charges can be defined by Gauss' law. The field tensor $A^{(k)}$ couples to a $(k-1)-D$ brane which has coordinates $X^\mu(\sigma_1,\cdots,\sigma_{k-1},\tau)$. Let $J^{(k)}$ be the k-form tangent to the brane. This acts as the source for the RR-field given by

$$dF^{(k+1)} = \mu_{k-1} * J^{(k)} \tag{11}$$

where μ_{k-1} is the charge of the $(k-1)$-brane. We can define 'electric' RR-charge by

$$e_k = \int_{S^{D-K-1}} *F_{(k+1)} \tag{12}$$

and a dual 'magnetic' RR-charge by its coupling to a $D-k-3$-brane by

$$g_{D-k-2} = \int_{S^{k+1}} F_{k+1}. \tag{13}$$

The equivalent Dirac quantization condition is

$$e_k \quad g_{D-k-2} = 2\pi n \tag{14}$$

where n is an integer. It turns out that in string theory $n = 1$ for D-branes and the charges e_k and g_p are dimensionless only for $D = 2(k+1)$. Thus

$D = 4$	$k = 1$	0-brane (particle)
$D = 6$	$k = 2$	1-brane (string)
$D = 10$	$k = 4$	3-brane

In the presence of a background gauge field living on the k-brane with field tensor $F = F_{\mu\nu} dx^\mu dx^\nu$, the coupling is given by

$$dF^{(k)} = \mu_{k-1} \mathrm{tr} \quad \mathrm{e}^{F/2\pi} * J^{(k)} \tag{15}$$

which comes from a term in the D-brane action of the form

$$\mu_{k-1} \int_{\Sigma_k} c \wedge \mathrm{tr}\, \mathrm{e}^{F/2\pi} \tag{16}$$

where c is the differential form associated with the sum of the RR-fields. For Type II A theory, for instance,

$$c = A^{(1)}_\mu dx^\mu + A^{(3)}_{\mu\nu\lambda} dx^\mu dx^\nu dx^\lambda$$

and from eq(16) we have tr $\displaystyle\int_{\Sigma^5} A^{(1)} \wedge F \wedge F$ on a 4-brane so that the $F \wedge F$ term couples to the field $A^{(1)}$ giving a 0-brane charge equal to $\displaystyle\left(\int_{\Sigma^4} F \wedge F\right) \int_{\Sigma^1} A^{(1)}$. In other words, by having a non-trivial background gauge field in the 4-brane, we have effectively created a 0-brane embedded within the 4-brane carrying the right charge to couple to the RR field $A^{(1)}_\mu$.

There is a symmetry in Type II strings which is a $U(1)$ gauge symmetry $\delta F = d\Lambda$. To preserve this symmetry we need to modify F to $F + B$, where $B_{\mu\nu}$ is the NS-NS field. The complete action for the single $(k+1)$-form gauge field $A^{(k+1)}$ is given by

$$\begin{aligned} S &= \frac{1}{2} \int *F^{(k+2)} \wedge F^{(k+2)} d^{10}x + \mu_k \int_\Sigma c \wedge \mathrm{tr} \quad \mathrm{e}^{\frac{F+B}{2\pi}} \\ &\quad + i\mu_k \int_\Sigma A^{(k+1)}_{\mu_1 \cdots \mu_{k+1}} \partial_1 X^{[\mu_1} \cdots \partial_{k+1} X^{\mu_k+1]} d^{k+1}\sigma \end{aligned}$$

where Σ is the k-brane world volume. The charges satisfy the Dirac quantization condition

$$\mu_{6-k}\mu_k = 2\pi.$$