Bosonic String

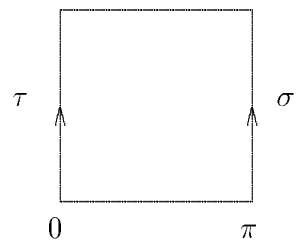
Let $X^\mu(\sigma,\tau)$ be the coordinates of the string in D-dimension with world volume M. Under conformal gauge, we have

$$S = \frac{1}{2\pi}\int_M d^2\sigma \partial^\alpha X^\mu \partial_\alpha X_\mu .$$

Reparametrization invariance is obtained by imposing the Virasoro conditions on the states space. To obtain the field equations we have

$$\delta S = -\frac{1}{\pi}\int_M d^2\sigma \delta X^\mu \partial^2 X_\mu + \frac{1}{2\pi}\int_{\partial M} d\tau \delta^\mu \partial_n X_\mu$$

where ∂_n is the derivative normal to ∂M. We shall choose the boundary conditions to set the boundary term in δS to zero. Consider Cartesian coordinates:



We shall choose the boundary conditions to set the boundary term to zero. Then the boundary term with $\partial_n = \dfrac{\partial}{\partial\sigma} = {}'$ is given by

$$\frac{1}{2\pi}\int \left[\delta X^\mu \delta X'_\mu(\pi,\tau) - \delta X^\mu X'_\mu(0,\tau)\right] d\tau .$$

This gives rise to several cases:

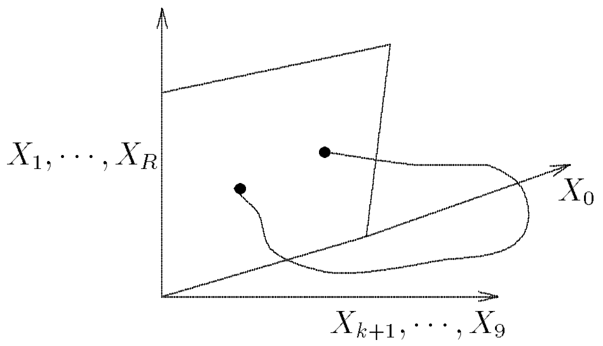
- Closed String
  $X_\mu(0,\pi) = X_\mu(0,\tau)$, periodic.
  Usually, in this case, one extends σ to $0 \le \sigma \le 2\pi$.

- Open String
  There are two possible boundary conditions:
  - Neumann (N) b.c.: $X'_{|0,\pi} = 0$
  - Dirichlet (D) b.c.: $\delta X_{|0,\pi} = 0$

  This gives four possible b.c.'s namely NN, ND, DN, DD.

For a D-k brane in 10 dimensions,

$$\begin{array}{lll} \text{(NN)} & X'_m(0) = 0 = X'_m(\pi), & m = 0,1,\cdots k \\ \text{(DD)} & X_i(0) = \text{constant} & \\ & X_i(0) = \text{constant}\ ' & m = k+1,\cdots 9 \end{array}$$



The ends of the open strings are free to move in the $(k+1)-$dimensional worldvolume of the k-brane. the $D-$brane is thus a rigid submanifold of spacetime.

For the usual open string which is part of the type I superstring, we have $k = 9$, i.e. all the components satisfy NN b.c.'s and the ends are free to move thorughout spacetime.

What is the consequence of the NN and DD b.c.'s on the fermions? As before, for the open strings, the overall sign is fixed by $\psi^\mu_+(\pi,\tau) = \psi^\mu_-(\pi,\tau)$. To preserve worldsheet supersymmetry, we have

$$\begin{array}{lll} \text{NN:} & \psi^\mu_+(0,\tau) = \pm\psi^\mu_-(0,\tau) & m = 0, 1, \cdots, k \\ \text{DD:} & \psi^i_+(0,\tau) = \mp\psi^i_-(0,\tau) & i = k+1, \cdots, 9 \end{array}$$

D-branes are BPS states and this is a consequence of the b.c's of the worldsheet fields. Of the two supercharges of Type II theory Q^L_α, Q^R_α, only the linear combination $Q^L_\alpha + \mathrm{e}^{i\phi_k} Q^R_\alpha$ is conserved for the open string, with ϕ_k being the phase coming from parity transformations for the transverse directions $X^i, i = k = k+1, \cdots, 9$. Since the D-branes have half the supersymmetry of Type II, they form BPS states.

Spectrum of states

For the open string with NN b.c.'s, we solve $\partial^2 X^\mu = 0$ to obtain

$$X^\mu(0,\tau) = x^\mu + p^\mu\tau + i\sum_{n\neq 0} \frac{\mathrm{e}^{in\tau}}{n} \alpha^\mu_n \cos(n\sigma)$$

with $[X^\mu, p^\nu] = i\eta^{\mu,\nu}$, $[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}$.

The Virasoro operator $(\alpha^\mu_0 = p^\mu)$,

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{m=1}^{\infty} \alpha_{-m}\alpha_m$$

imposes constraint on physical states

$$(L_0 - 1)|\text{phy¿} = 0$$

which yields the mass spectrum

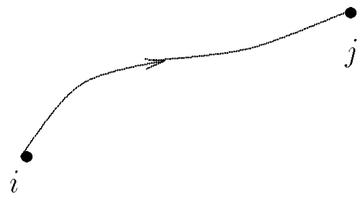
$$M^2 = -p^\mu p_\mu = \sum_{m=1}^{\infty} \alpha_{-m}\alpha_m - 2$$

Define $|0,k>_B$ by $\alpha_m|0,k>_B = 0, m > 0$, we then have massless state and tachyon vacua:

$$\begin{array}{rll} \text{tachyon:} & |0,k>_B, & M^2 = -1 \\ \text{photon:} & \alpha_{-1}^{\mu}|0,k>_B, & M^2 = 0 \end{array}$$

Chan-Paton factors

We can attach non-dynamical degrees of freedom to the ends of the open string.



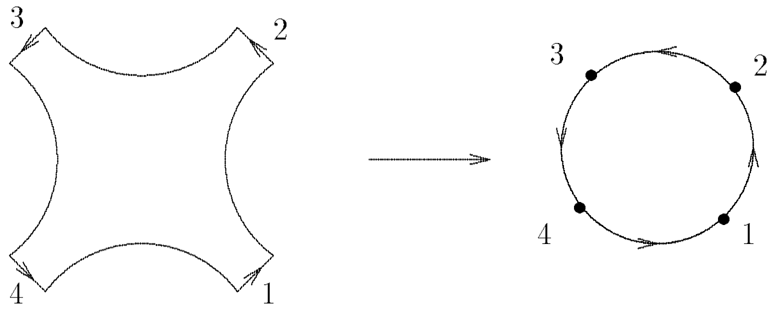
$i, j = 1, 2, \cdots, N$

String state is $|k, ij>$

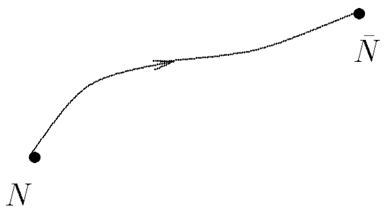
The $N \times N$ matrices λ_{ij}^a form a basis decompose the string wave function into an irrep called the Chan-Paton factors. We have

$$|k, a> = \sum_{ij} \lambda_{ij}^a |k, ij>$$

Unitary demands $\lambda^\dagger = \lambda$, so that λ is Hermitian. The scattering of two oriented strings appears as:



This diagram has a factor $\mathrm{tr}(\lambda^1\lambda^2\lambda^3\lambda^4)$. For an oriented string, one can think of the two ends carrying the fundamental N and $\bar{N}$ as:



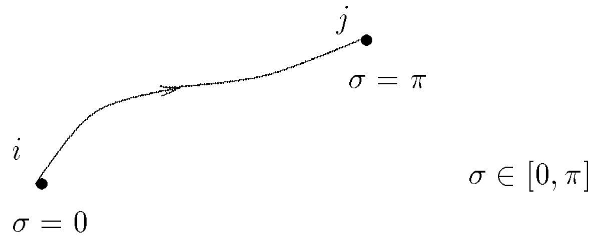
In order that the massless excitations of the open string can be precisely those of Yang-Mills gauge fields $N \otimes \bar{N} \sim \lambda^a$ since the fields have to be expressed in the adjoint represenation of the Lie group, G. There are only three solutions for consistent open strings given by:

$$
\begin{array}{rll}
G = U(N) & \bar{N} = N^* & \mathrm{orientable\ \ strings} \\
G = SO(N) & \bar{N} = N & \mathrm{non-orientable\ \ strings} \\
G = Sp(N) & \bar{N} = N & \mathrm{non-orientable\ \ strings}
\end{array}
$$

We will later show that in $D = 10$ only the $SO(32)$ case is consistent. Hence we have to analyze the nonorientable strings.

The D-brane interpretation is that there are exactly 32 degenerate 9-branes if the open superstring is to avoid tadpole singularities. The subscripts i, j in λ_{ij} indicates on which degenerate 9-branes the two endpoints are on.

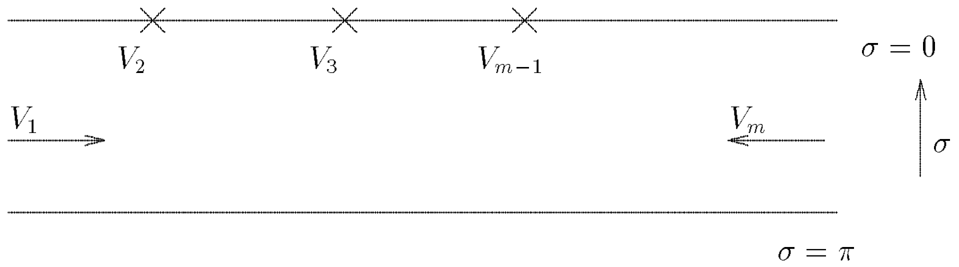
Non-oriented strings



For the case above in which the indices i and j are equivalent, we can associate a vertex operator $V_\Lambda(k, \tau)$ for every physical state Λ with momentum k^μ by

$$V_\Lambda(k, \tau) = \mathrm{e}^{i\tau L_0} V_\Lambda(k, 0) \mathrm{e}^{-i\tau L_0}.$$

This vertex operator describes the emission of the state Λ from the $\sigma = 0$ end of the open string. For example, for the string propagator $\Delta = (L_0 - 1)^{-1}$, the amplitdue $< 1|V_2(k_2)\Delta V_3(k_3)\Delta \cdots V_{M-1}(k_M)|M >$ is represented by:



For nonorientable open string, emission from $\sigma = 0$ edge should be symmteric with emissions from $\sigma = \pi$ edge. This entails the requirement that the vertex operators be built from $X(\sigma = \pi, \tau)$ instead of $X(\sigma = 0, \tau)$. To achieve this , we introduce the 'twist operator' Ω which maps σ to $\pi - \sigma$, so that

$$V(\sigma = \pi) = \Omega V(\sigma = 0)\Omega^{-1}.$$

Clearly, $\Omega^2 = 1$. Replacing $\sigma = 0$ by $\sigma = \pi$ for the open string can be achieved by a change of variables $\sigma \to \pi - \sigma = \sigma'$; hence

$$X(\sigma', \tau) \quad = \quad x^\mu + p^\mu \tau + i \sum_n{}' \frac{\mathrm{e}^{-in\tau}}{2n} \alpha_n^\mu \left[\mathrm{e}^{-in\pi} \mathrm{e}^{i\sigma} \ + \mathrm{e}^{+in\pi} \mathrm{e}^{-i\sigma} \right]$$

$$
\begin{aligned}
&= \Omega X(\sigma,\tau) \\
\Rightarrow \Omega\alpha_n &= (-1)^n\alpha_n, \Omega x^\mu = x^\mu \\
\Omega p^\mu &= p^\mu, \Omega(\sigma) = \pi - \sigma
\end{aligned}
$$

Hence $\Omega = (-1)^{N_B}$, where N_B is a bosonic number operator with $[N_B, \alpha_n] = n\alpha_n$. The vacuum has $\Omega|0,k>_B = +|0,k>_B$. In terms of these operators, for the photon, we have

$$\Omega\alpha^\mu_{-1}|0,k> = -\alpha^\mu_{-1}|o,k>$$

The world sheet parity operator acts non-trivially on the Chan-Paton factors.

$$\Omega\lambda_{ij}|k,ij> = \lambda'_{ij}|k,ij>$$

where $\lambda' = M\lambda^T M^{-1}$. Since $\Omega^2 = 1$, we have in general

$$\lambda^a = MM^{-T}\lambda^{aT}M^TM^{-1}.$$

Since λ^a must form a complete set by CPT, we have by Schur's lemma $M^TM^{-1} = \pm 1_N$. There are two possibilities:

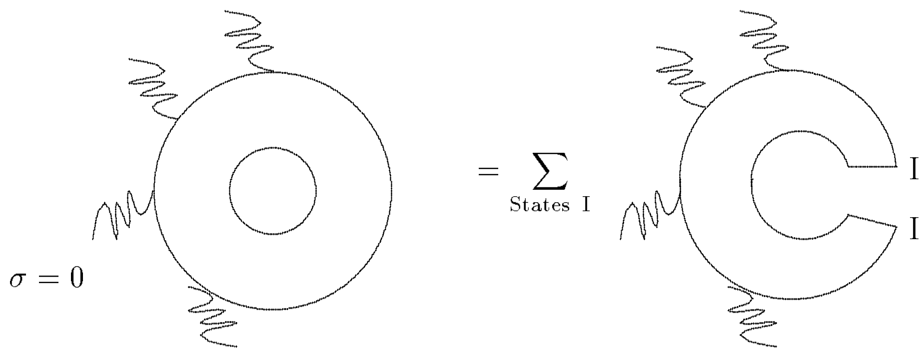
$$
\begin{array}{rl}
1. & M^T = +M = I_N \\
\text{tachyon:} & \Omega\lambda_{ij}|0,k,ij> = \lambda^T_{ij}|0,k,ij> \\
\text{photon:} & \Omega\lambda_{ij}\alpha^\mu_{-1}|0,k,ij> = -\lambda^T_{ij}\alpha^\mu_{-1}|0,k,ij> \\
 & \\
2. & M^T = -M = i\begin{pmatrix} 0 & I_{N/2} \\ I_{N/2} & 0 \end{pmatrix} \equiv J \\
\text{tachyon:} & \Omega\lambda_{ij}|0,k,ij> = (J\lambda^TJ)_{ij}|0,k,ij> \\
\text{photon:} & \Omega\lambda_{ij}\alpha^\mu_{-1}|0,k,ij> = -(J\lambda^TJ)_{ij}\alpha^\mu_{-1}|0,k,ij>
\end{array}
$$

For non-orientable strings, invariance under Ω is a symmetry of the action and all amplitudes. To obatin a Hilbert space of states invariant under Ω, w etake the oriented string spectrum and project out states with the projector $P = \frac{1}{2}(1+\Omega)$, with $P^2 = P$.

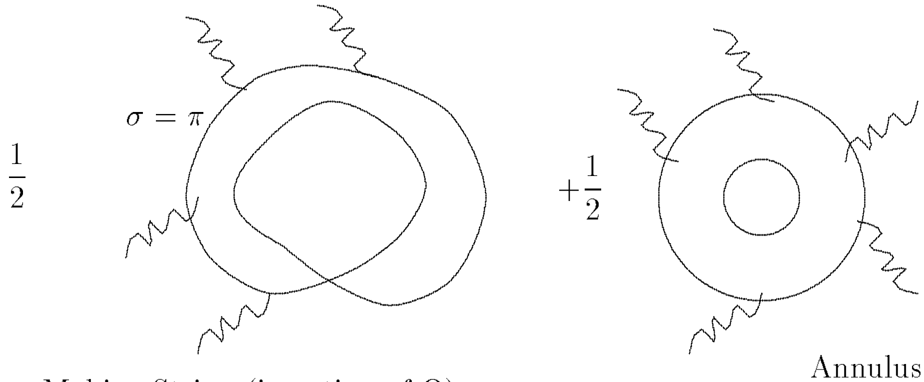
For the open string with Chan-Paton charges P will project out the tachyon and retain the phton if we demand for:

Case 1	$\lambda^T = -\lambda$	i.e. $SO(N)$ gauge group
Case 2	$J\lambda^TJ = -\lambda$	i.e. $Sp(N)$ gauge group

Nonorientable worldsheets arise in the following way. For every oriented string amplitude with vertex operators, the one loop daigram gives a sum over a complete set of intermediate states I.



But when the projector $\frac{1}{2}(1+\Omega)$ is inserted into the trace it switches the emission operators from $\sigma = 0$ to $\sigma = \pi$.



Mobius String (insertion of Ω)

Annulus

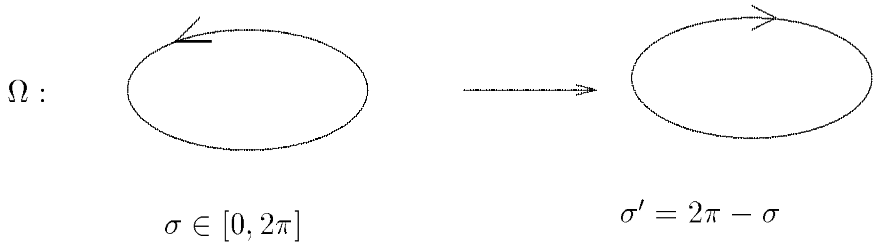
Type I Superstring

Type I is constructed from open strings (both oriented and non-oriented) as well as the closed string sector. The Hilbert space $\mathcal{F}_{open}$ is Ω-symmetric, i.e.

$$\Omega \mathcal{F}_{open} = \mathcal{F}_{open}$$

In GSO projected RNS superstring, the Type I superstring sector yields an $N = 1$ supersymmetric Yang-Mill multiplet in $D = 10$ with $SO(32)$ being the only consistent gauge group. Thus

$$\left.\begin{array}{rll} \mathrm{NS}: & 8 \times 496 & A_\mu \\ \mathrm{R}: & 8 \times 496 & \psi^a \end{array}\right\} \mathrm{space-time} \quad \mathrm{susy}$$



For the closed string sector, the action of Ω is to switch the left and right movers. To be symmetric under Ω, the left and right movers must have the same parity. Hence, we must start with IIB and retain only states invariant under Ω, i.e. $\mathcal{F}_{open,\, closed\ sector} = \mathcal{F}_{IIB}/\Omega$.

On making this projection, we obtain the Type I closed string sector with space-time supersymmetric spectrum given by

NS-NS	bosons	$G_{\mu\nu}$	35
		Φ	1
R-R	bosons	$A^{(2)}_{\mu\nu}$	28
		Total:	64
NS-R	fermions	χ^α_μ	56
R-NS		λ_α	8
		Total:	64

From the RR-sector, we see that Type I couples to a 1-D brane, which is dual to a 5-D brane. It will turn out that it also couples to a 9-D brane.