T-Duality

For the closed string, we have

$$\partial^2 X^\mu = 0$$

with the boundary condition $X^\mu(\sigma + 2\pi) = X^\mu(\sigma, \tau)$ and yields normal mode expansion.

$$\begin{aligned}
X^\mu(\sigma,\tau) &= x^\mu + p^\mu \tau + \frac{i}{2}{\sum_n}' \frac{1}{n}(\alpha_{n_i} \mathrm{e}^{-in(\tau-\sigma)} + \tilde{\alpha}_{n_i} \mathrm{e}^{-in(\tau+\sigma)}) \\
&= x_L^\mu + x_R^\mu + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu(\tau-\sigma) + \sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_0^\mu(\tau+\sigma) + \text{oscillators} \\
&= X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma)
\end{aligned}$$

Note

$$p^\mu = \sqrt{\frac{1}{2\alpha'}}(\alpha_0^\mu + \tilde{\alpha_0^\mu})$$

Under $\sigma \to \sigma + 2\pi, X^\mu(\sigma,\tau)$ changes by $\sqrt{\frac{\alpha'}{2}}(\tilde{\alpha}_0^\mu - \alpha_0^\mu)2\pi$. For non-compact spatial dimension X^μ is single-valued and hence

$$\tilde{\alpha}_0^\mu = \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$$

Suppose X^{25} in a circle with radius R. Then, under a shift of $\sigma \to \sigma + 2\pi$, we can have

$$X^{25}(\sigma + 2\pi, \tau) = X^{25}(\sigma,\tau) + 2\pi R w$$

where w is the winding number $\in Z$.

Hence

$$\begin{aligned}
\alpha_0^{25} + \tilde{\alpha}_0^{25} &= \frac{2n}{R}\sqrt{\frac{\alpha'}{2}} \\
\alpha_0^{25} - \tilde{\alpha}_0^{25} &= wR\sqrt{\frac{2}{\alpha'}}
\end{aligned}$$

since $p^{25} = \dfrac{n}{R}$. Furthermore, we have

$$\begin{aligned}
\alpha_0^{25} &= (\frac{n}{R} + \frac{wR}{\alpha'})\sqrt{\frac{\alpha'}{2}} \\
\tilde{\alpha}_0^{25} &= (\frac{n}{R} - \frac{wR}{\alpha'})\sqrt{\frac{\alpha'}{2}}
\end{aligned}$$

The mass spectrum is

$$\begin{aligned}
M^2 = -p^\mu p_\mu &= \frac{2}{\alpha'}(\alpha_0^{25})^2 + \frac{4}{\alpha'}(L_0 - 1) \\
&= \frac{2}{\alpha'}(\tilde{\alpha}_0^{25})^2 + \frac{4}{\alpha'}(\tilde{L}_0 - 1)
\end{aligned}$$

Note

$$M^2(n,w,R) = M^2(w,n,\frac{\alpha'}{R})$$

i.e. identical spectrum by $n \leftrightarrow w, R \leftrightarrow \dfrac{\alpha'}{R}$
Under this transformation

$$\begin{aligned} \alpha_0^{25} &\rightarrow \alpha_0^{25} \\ \tilde{\alpha}_0^{25} &\rightarrow -\tilde{\alpha}_0^{25} \end{aligned}$$

In other words, the T-dual theory is described by a parity transformation on left-movers, i.e.

$$T_{25}[X^{25}] = X'^{25}(\sigma,\tau) = X_R^{25}(\tau-\sigma) - X_L^{25}(\tau+\sigma)$$

T-duality is an exact symmetry of string theory. In general, for k-compact directions we have $T_{\mu_1,\cdots,\mu_k}$.

The effective coupling in 25^{th}-dimension is $\mathrm{e}^{\phi}R^{-\frac{1}{2}}$ (ϕ: dilaton). hence in the T-dual theory,

$$\begin{aligned} \mathrm{e}^{\phi'}R'^{-1/2} &= \mathrm{e}^{\phi}R^{1/2} \\ \text{or,} \qquad \mathrm{e}^{\phi'} &= \sqrt{\alpha'}\mathrm{e}^{\phi}R^{-1} \end{aligned}$$

Open Strings and D-branes

Recall with NN-conditions, we have

$$\begin{aligned} X^{\mu}(\sigma,\tau) &= x^{\mu} + p^{\mu}\tau(2\alpha') + \frac{i}{2}{\sum_n}' \frac{\alpha_n^{\mu}}{n}(\mathrm{e}^{-in(\tau-\sigma)} + \mathrm{e}^{-in(\tau+\sigma)} \\ &= X_R^{\mu}(\tau-\sigma) + X_L^{\mu}(\tau+\sigma). \end{aligned}$$

For the open string, left and right movers are refelcted at the ends and form standing waves. We have

$$\begin{aligned} X_R^{\mu}(\tau-\sigma) &= \frac{x^{\mu}}{2} + c + \alpha'(\tau+\sigma)p^{\mu} + \frac{i}{2}\sum_n \frac{\mathrm{e}^{-in(\tau-\sigma)}}{n}\alpha_n^{\mu} \\ X_L^{\mu}(\tau-\sigma) &= \frac{x^{\mu}}{2} - c + \alpha'(\tau-\sigma)p^{\mu} + \frac{i}{2}\sum_n \frac{\mathrm{e}^{-in(\tau+\sigma)}}{n}\alpha_n^{\mu} \end{aligned}$$

Suppose X^{25} is compact with radius R. This implies $p^{25} = \dfrac{n}{R}$. For closed strings, T-duality was a symmetry because we could interchange windings of the closed string in X^{25} namely w with the KK-modes dues to the comapctness of X^{25}. For the open string there is no winding number. So how do we obtain T-duality?

As before, we define the T-dual string by

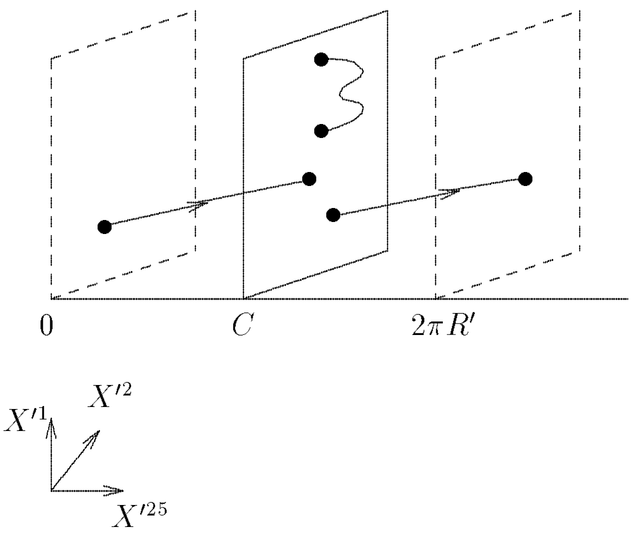
$$T_{25}[X^{25}] = X'(\sigma,\tau) \equiv X_R^{25}(\tau-\sigma) - X_L^{25}(\tau-\sigma)$$

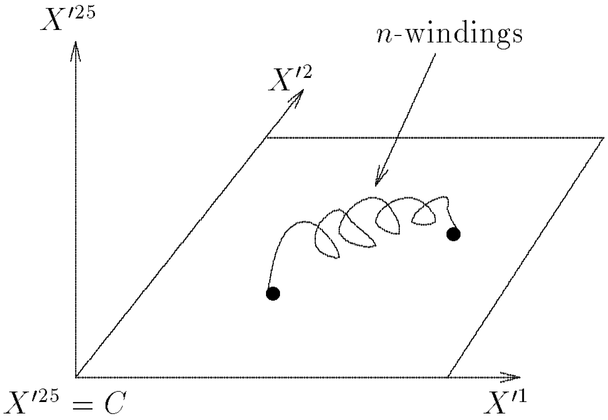
or, $X'^{25}(\sigma,\tau) = 2c + 2\alpha' p^{25}\sigma - {\sum_n}' \dfrac{\mathrm{e}^{-in\tau}}{n}\sin(n\sigma)\alpha_n^{25}$. Note at the boundaries $\sigma = 0, \pi$, the oscillators terms disappear and since there is no other dependence on τ, the boundaries do not move! By T-duality we have switched the NN-condition $\frac{\partial X^{25}}{\partial \sigma}|_{0,\pi} = 0$ to DD-condition $\frac{\partial X^{25}}{\partial \tau}|_{0,\pi} = 0$!. Thus

$$
\begin{aligned}
X'25(0,\tau) &= 2c \\
X'25(\pi,\tau) &= 2c + 2\pi\alpha' p^{25} \\
&= 2c + 2\pi n R'(*)
\end{aligned}
$$

From $(*)$, we see that in the dual theory with $R' = \alpha'/R$, the X'^{25} ends are fixed, i.e. $X'^{25}(0,\tau) = X'^{25}(\pi,\tau)$ mod $2\pi R'$.

Since the X'^{25} boundaries cannot move, it is meaningful if the X'^{25} coordinates winds about the 25^{th} direction since the DD-conditions prevent it from un-winding.





We see that open strings have T-duality symmetry in a manner very different from closed strings. Instead of interchanging winding number with KK-momenta as is the case for closed strings, for open strings the $DD - b.c.'s$ in effect switch the KK-momenta into a winding number in the dual theory!

$R \to 0$ Limit

For closed strips, as $R \to 0$ only the $n = 0 KK$ mode survives and we have

$$
\begin{aligned}
\lim_{R\to 0} M^2 &= \frac{2}{\alpha'}(\frac{n}{R} + \frac{wR}{\alpha'})^2 + \frac{4}{\alpha'}(L_0 - 1) \\
&\to \frac{2}{\alpha'^3}(wR)^2 + \frac{4}{\alpha'}(L_0 - 1) \\
&= \frac{2}{\alpha'^3}W^2 + \frac{4}{\alpha'}(L_0 - 1)
\end{aligned}
$$

where $W \in [-\infty, +\infty]$ is the effect on the mass spectrum due to the compact direction. This effect is a purely string effect since as $R \to 0$ the closed string finds it energetically more and more favourable to wind around a small R and winding number w becomes arbitrarily large yielding real continuous variable $W = wR$: a new continuum of states labelled by W.

For open strings, as $R \to 0$, the KK momenta is $n = 0$ and hence in the dual picture winding number is also zero. Unlike the closed string, there is no new continuum of states. Recall

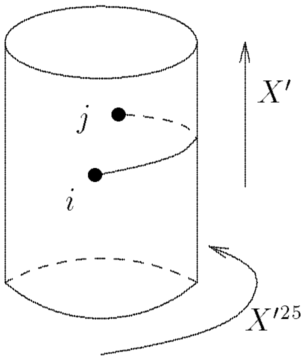
$$X'^{25}(\pi) - X'^{25}(0) = 2\pi n R' \to 0$$

That is, the open string is only free to move in $(D-1)$-dimensions as $R \to 0$ unlike closed strings which continue to vibrate in D-dimensions even as $R \to$.

Wilson Lines, Open Strings and T-Duality

Recall for open strings we have non Abelian gauge fields $A^a_\mu(x)$ in the expansion for the open string.

$$A^a_\mu \lambda^a_{ij} \alpha^\mu_{-1} |k, ij>$$



For compact direction X^{25} we can consider the gauge field component $A^{25}_{ij} = A^{a25}\lambda^a_{ij}$ to be a background gauge field. Consider a $U(N)$ oriented string with end points carrying N and $\bar{N}$ irreps. One can consider Wilson line by

$$P_{\mathrm{Adj}} \mathrm{e}^{i \int_{X^{25}(0)}^{X^{25}(\pi)} A^{25} dx^{25}}$$

and the background field in fundamental irrep

$$A^{25} = \frac{1}{2\pi R} \begin{pmatrix} \theta_1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \theta_N \end{pmatrix} = -i\Lambda^{-1} \frac{\partial \Lambda}{\partial x^{25}}$$

which breaks $U(N)$ to $U(1)^N$.

We have in the fundamental representation,

$$\Lambda = \mathrm{diag}(\mathrm{e}^{\frac{iX^{25}\theta_1}{2\pi R}}, \cdots, \mathrm{e}^{\frac{iX^{25}\theta_N}{2\pi R}})$$

We can set A^{25} to zero by a gauge transformation Λ. We also have simultaneously gauge transformation on the state by

$$\begin{aligned} |ij> &\rightarrow V_{\alpha i}^{-1}(X^{25}(0))V_{\beta j}^{-1}(X^{25}(0))|ij> \\ &= \mathrm{e}^{\frac{i}{2\pi R}(\theta_j X^{25}(\pi)-\theta_i X^{25}(0))}|ij> \end{aligned}$$

For the string state (V = vertex operator)

$$|\mathrm{string}>= \mathrm{e}^{ip^{25}X^{25}}|ij>$$

under $X^{25} \rightarrow X^{25}+2\pi R$, $|ij> \rightarrow \mathrm{e}^{i(\theta_j-\theta_i)}|ij>$. Hence $p^{25} = \dfrac{1}{2\pi R}(\theta_j-\theta_i+2n\pi R)$. In other words, a Wilson line formed by breaking $A^{25} \rightarrow < A^{25} > = -i\Lambda^{-1}\partial_{25}\Lambda$ is effectively imparting fractional momenta to the open string.

Recall under T duality

$$\begin{aligned} X'^{25}(\pi) - X'^{25}(0) &= 2\pi\alpha' p^{25} \\ &= \frac{\alpha'}{R}(2\pi n + \theta_j - \theta_i) \end{aligned}$$

The state of the string at $\sigma = 0$ should depend only on i and at $\sigma = \pi$ on only j (otherwise we would violate locality).

$$\begin{aligned} X'^{25}(\pi) &= 2\pi n R' + \theta_j R' \\ X'^{25}(0) &= \theta_i R' \end{aligned}$$

In other words, for $\theta_i \neq \theta_j$, in the dual spacetime, the open string endpoint for state i is located on a D-brane placed at $\theta_i R'$ along X'^{25} and the other endpoint is located on a D-brane at $\theta_j R'$ (both ends located modulo $2\pi R'$).