NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2016-2017

MA1101R Linear Algebra I

November 2016 — Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **Six (6)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions.

3. This is a **closed book** examination but each candidate is allowed to bring in **TWO (2)** double-sided A4-sized handwritten helpsheets.

4. Calculators can be used. However, various steps in the calculations should be laid out systematically.

5. Write down your matriculation/registration number on the cover page of each answer book used.

Question 1 [14 Marks]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & k & 1 & k \\ k & k & 2 & 0 \\ k & 0 & k & 0 \end{pmatrix}$ where k is a constant.

(a) Use Gauss-Jordan Elimination to reduce $\boldsymbol{A}$ to the reduced row-echelon form. (Write down the elementary row operations clearly.)

(b) Find a basis for the nullspace space of $\boldsymbol{A}$.

(*Warning*: The value of k will affect your answers.)

Question 2 [12 Marks]

Let $V = \{\, (a+b-2c,\ 2b-c,\ 3c+d,\ a+3b+d) \mid a,b,c,d \in \mathbb{R}\,\}$.

(a) Show that V is a subspace of $\mathbb{R}^4$.

(b) Find a basis for V and determine the dimension of V.

(c) Let $W = \{\, (1+a+b-2c,\ 2b-c,\ 3c+d,\ 1+a+3b+d) \mid a,b,c,d \in \mathbb{R}\,\}$.

(i) Is the zero vector contained in W? Justify your answer.

(ii) Is W a subspace of $\mathbb{R}^4$?

Question 3 [18 Marks]

Let $\boldsymbol{B} = \begin{pmatrix} 4 & 0 & 2 & -2 \\ 0 & 4 & -2 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}$.

(a) Find an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ so that $\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} = \boldsymbol{D}$.

(b) Write down a matrix $\boldsymbol{C}$ such that $\boldsymbol{C}^2 = \boldsymbol{B}$.

(You can express your answer in the form $\boldsymbol{P}\boldsymbol{X}\boldsymbol{P}^{-1}$ where $\boldsymbol{X}$ is a 4×4 matrix and $\boldsymbol{P}$ is the invertible matrix obtained in (a).)

Question 4 [22 Marks]

(All vectors in this question are written as column vectors.)

Let $\boldsymbol{u_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{u_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{u_3} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, and $\boldsymbol{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\boldsymbol{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(a) (i) Find the reduced row-echelon form of the following 3×6 matrix

$$\begin{pmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{u_3} & \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \end{pmatrix}.$$

(ii) Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$. You can assume that S is a basis for $\mathbb{R}^3$.

Write down the coordinate vectors $(\boldsymbol{e_1})_S$, $(\boldsymbol{e_2})_S$, $(\boldsymbol{e_3})_S$.

(b) Define a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T(\boldsymbol{x}) = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} \quad \text{for } \boldsymbol{x} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + c_3\boldsymbol{u_3} \in \mathbb{R}^3.$$

(i) Find the standard matrix for T.

(ii) Determine the rank and nullity of T.

(iii) Explain why $T(\boldsymbol{x})$ is the orthogonal projection of $\boldsymbol{x}$ onto $V = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$.

Question 5 [17 Marks]

(All vectors in this question are written as column vectors.)

Let $\boldsymbol{A}$ be a square matrix of order n.

(a) Let $E = \{\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_n}\}$ be the standard basis for $\mathbb{R}^n$.

Show that $\boldsymbol{A}\boldsymbol{e_j} =$ the jth column of $\boldsymbol{A}$.

(b) Suppose $\boldsymbol{A}^m = \boldsymbol{0}$ and $\boldsymbol{A}^{m-1} \neq \boldsymbol{0}$ for some integer $m \geq 2$.

(i) Show that there exists at least one vector $\boldsymbol{u} \in \mathbb{R}^n$ such that $\boldsymbol{A}^{m-1}\boldsymbol{u} \neq \boldsymbol{0}$.

(ii) Show that $\{\boldsymbol{u}, \boldsymbol{A}\boldsymbol{u}, \ldots, \boldsymbol{A}^{m-1}\boldsymbol{u}\}$ is linearly independent where $\boldsymbol{u}$ is the vector obtained in Part (i).

(c) Prove that if $\boldsymbol{A}^{n+1} = \boldsymbol{0}$, then $\boldsymbol{A}^n = \boldsymbol{0}$.

Question 6 [17 Marks]

(All vectors in this question are written as column vectors.)

(a) Let $\boldsymbol{B}$ be a 2×2 symmetric matrix and let $\boldsymbol{u}$, $\boldsymbol{v}$ be two eigenvectors of $\boldsymbol{B}$ associated with the eigenvalues λ and μ respectively.

(i) Show that if $\lambda \neq \mu$, then $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

(ii) Suppose $\boldsymbol{u} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $\lambda = 1$ and $\mu = 3$.

Find $\boldsymbol{B}$. (Hint: $\{\boldsymbol{u}, \boldsymbol{v}\}$ is an orthonormal basis for $\mathbb{R}^2$.)

(b) Let $\boldsymbol{C}$ be a symmetric matrix of order n with a characteristic polynomial

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

Prove that for any nonzero vector $\boldsymbol{x} \in \mathbb{R}^n$, $\lambda_1 \leq \dfrac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}} \leq \lambda_n$.

(Hint for (a)(i) and (b): For $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v}$.)

[END OF PAPER]