National University of Singapore
Department of Mathematics

Semester 1, 2017/18

MA1101R Linear Algebra I

November 2017 — Time allowed: 2 hours

Student Number: ____________________

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of 6 questions, for a total of 80 points. Excluding the cover page, there are 12 printed pages.
2. Answer all 6 questions.
3. This is a closed book examination but you are allowed to bring in one A4-size and double-sided helpsheet.
4. You can use any kind of calculators (except devices which can be used for communication and/or web-surfing). However, various steps in the calculations should be laid out systematically.
5. Write down your student number on the cover page of this booklet.
6. Write your answers in the space below each question. This booklet will be collected at the end of the examination.
7. The left-hand pages can be used for rough work.

Question	Points	Score
1	11	
2	8	
3	9	
4	18	
5	17	
6	17	
Total:	80	

1. (11 points) Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$.

(a) Use the **Gauss-Jordan Elimination** to reduce $\boldsymbol{A}$ to its reduced row-echelon form. (Write down the steps of your computations.)

(b) Write down a basis for the row space of $\boldsymbol{A}$.

(c) Write down a basis for the column space of $\boldsymbol{A}$.

(d) Write down a basis for the nullspace of $\boldsymbol{A}$.

2. (8 points) Let $V = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ where $\boldsymbol{u_1} = (1,1,0,0)$, $\boldsymbol{u_2} = (1,1,-1,-1)$ and $\boldsymbol{u_3} = (1,a,1,a)$ where a is an unknown constant.

   Apply the Gram-Schmidt Process to $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ to obtain an orthonormal basis for V.

   (Warning: The value of a may affect your answer.)

3. (9 points) Let W be a vector space with a basis $S = \{\, \boldsymbol{v_1},\ \boldsymbol{v_2},\ \boldsymbol{v_3} \,\}$. Let $T = \{\, \boldsymbol{w_1},\ \boldsymbol{w_2},\ \boldsymbol{w_3} \,\}$ where

$$\boldsymbol{w_1} = \boldsymbol{v_1} + 2\boldsymbol{v_2}, \quad \boldsymbol{w_2} = \boldsymbol{v_2} + 2\boldsymbol{v_3} \quad \text{and} \quad \boldsymbol{w_3} = \boldsymbol{v_3}.$$

(a) Show that T is a basis for W.

(b) Find the transition matrix from S to T.

4. (18 points) Let $\boldsymbol{B} = \begin{pmatrix} -2 & 0 & -2 & 1 \\ -1 & -1 & -2 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

(a) Find the characteristic polynomial of $\boldsymbol{B}$ and verify that the eigenvalues of $\boldsymbol{B}$ are -1 and 0.

(b) Find a basis for the eigenspace E_{-1} of $\boldsymbol{B}$.

(c) Find a basis for the eigenspace E_0 of $\boldsymbol{B}$.

(d) Write down an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} = \boldsymbol{D}$.

(e) Find $\boldsymbol{B}^{1101}$.

5. (17 points) Let $\boldsymbol{C}$ be a square matrix.

(a) Show that the nullspace of $\boldsymbol{C}$ is a subset of the nullspace of $\boldsymbol{C}^2$.

(b) If $\text{rank}(\boldsymbol{C}^2) = \text{rank}(\boldsymbol{C})$, show that the nullspace of $\boldsymbol{C}^2$ is equal to the nullspace of $\boldsymbol{C}$.

(c) Give an example of a 2×2 matrix $\boldsymbol{C}$ with $\operatorname{rank}(\boldsymbol{C}^2) = \operatorname{rank}(\boldsymbol{C})$.

(d) Give an example of a 2×2 matrix $\boldsymbol{C}$ with $\operatorname{rank}(\boldsymbol{C}^2) < \operatorname{rank}(\boldsymbol{C})$.

(e) Can $\operatorname{rank}(\boldsymbol{C}^2) > \operatorname{rank}(\boldsymbol{C})$? Why?

6. (17 points) Let $\boldsymbol{A}$ be an $n \times n$ matrix.

For each $\lambda \in \mathbb{R}$, we define a linear transformation $T_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$T_\lambda(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u} - \lambda\boldsymbol{u} \quad \text{for} \quad \boldsymbol{u} \in \mathbb{R}^n.$$

(a) Write down the standard matrix for T_λ.

(b) For any $\lambda, \mu \in \mathbb{R}$, show that

$$(\boldsymbol{A} - \lambda\boldsymbol{I})(\boldsymbol{A} - \mu\boldsymbol{I}) = (\boldsymbol{A} - \mu\boldsymbol{I})(\boldsymbol{A} - \lambda\boldsymbol{I}).$$

(c) Suppose $\boldsymbol{A}$ is diagonalizable and the eigenvalues of $\boldsymbol{A}$ are λ_1, λ_2, ..., λ_k.

(i) If $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$, say, $\boldsymbol{A}\boldsymbol{v} = \lambda_i \boldsymbol{v}$ for some i, show that $(\boldsymbol{A} - \lambda_1 \boldsymbol{I})(\boldsymbol{A} - \lambda_2 \boldsymbol{I}) \cdots (\boldsymbol{A} - \lambda_k \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}$.

(Hint: First, show that $(\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}$ and then use the result in part (b).)

(ii) Define $S = T_{\lambda_1} \circ T_{\lambda_2} \circ \cdots \circ T_{\lambda_k}$. Prove that S is the zero transformation.