Question 1:

(a) $$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \xrightarrow[R_3-R_1]{R_2+R_1} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \xrightarrow[R_4-R_2]{R_1-R_2} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Basis for row space: $\left\{ \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}$

(c)

Basis for column space: $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

(d) $$\begin{cases} x_1 - x_4 = 0 \\ x_3 + x_4 = 0 \\ x_5 = 0 \end{cases}$$

Let $x_2 = s$, $x_4 = t$, for some $s, t \in \mathbb{R}$. Then $x_1 = t$ and $x_3 = -t$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} t \\ s \\ -t \\ t \\ 0 \end{pmatrix} = s \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

Basis for nullspace: $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Question 2:

$$
\begin{aligned}
\boldsymbol{v_1} &= \boldsymbol{u_1} \\
\boldsymbol{v_2} &= \boldsymbol{u_2} - \frac{\boldsymbol{u_2} \cdot \boldsymbol{v_1}}{||\boldsymbol{v_1}||^2}\boldsymbol{v_1} \\
&= \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix} - \frac{2}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} \\
\boldsymbol{v_3} &= \boldsymbol{u_3} - \frac{\boldsymbol{u_3} \cdot \boldsymbol{v_1}}{||\boldsymbol{v_1}||^2}\boldsymbol{v_1} - \frac{\boldsymbol{u_3} \cdot \boldsymbol{v_2}}{||\boldsymbol{v_2}||^2}\boldsymbol{v_2} \\
&= \begin{pmatrix} 1 & a & 1 & a \end{pmatrix} - \frac{1+a}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} - \frac{-1-a}{2}\begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1-a}{2} & \frac{a-1}{2} & \frac{1-a}{2} & \frac{a-1}{2} \end{pmatrix} \\
&= \frac{a-1}{2}\begin{pmatrix} -1 & 1 & -1 & 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{w_1} &= \frac{1}{||\boldsymbol{v_1}||}\boldsymbol{v_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \\
\boldsymbol{w_2} &= \frac{1}{||\boldsymbol{v_2}||}\boldsymbol{v_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix}
\end{aligned}
$$

If $a = 1$, $\dim(V) = 2$. Orthonormal basis for V: $\{\boldsymbol{w_1}, \boldsymbol{w_2}\}$.

Otherwise, suppose $a \neq 1$.

$$
\boldsymbol{w_3} = \frac{1}{a-1}\boldsymbol{v_3} = \frac{1}{2}\begin{pmatrix} -1 & 1 & -1 & 1 \end{pmatrix}
$$

Orthonormal basis for V: $\{\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}\}$.

Question 3:

$$\boldsymbol{v_3} = \boldsymbol{w_3} \qquad [\boldsymbol{v_3}]_T = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$$

$$\begin{aligned} \boldsymbol{v_2} &= \boldsymbol{w_2} - 2\boldsymbol{v_3} \\ &= \boldsymbol{w_2} - 2\boldsymbol{w_3} \end{aligned} \qquad [\boldsymbol{v_2}]_T = \begin{pmatrix} 0 & 1 & -2 \end{pmatrix}^T$$

$$\begin{aligned} \boldsymbol{v_1} &= \boldsymbol{w_1} - 2\boldsymbol{v_2} \\ &= \boldsymbol{w_1} - 2\boldsymbol{w_2} + 4\boldsymbol{w_3} \end{aligned} \qquad [\boldsymbol{v_1}]_T = \begin{pmatrix} 1 & -2 & 4 \end{pmatrix}^T$$

The transition matrix, P, from S to T is given by

$$\begin{aligned} P &= \begin{pmatrix} [\boldsymbol{v_1}]_T & [\boldsymbol{v_2}]_T & [\boldsymbol{v_3}]_T \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix} \end{aligned}$$

Since P is invertible ($|P| \neq 0$), T is linearly independent. Furthermore, $\text{span}(T) \subseteq W = \text{span}(S)$ and $|T| = |S|$. By Theorem 3.6.9, $\text{span}(T) = \text{span}(S)$ and thus T is a basis for W.

Question 4:

(a) $$\begin{aligned} p(x) &= |xI - B| \\ &= \begin{vmatrix} x+2 & 0 & 2 & -1 \\ 1 & x+1 & 2 & -1 \\ -1 & 0 & x-1 & 1 \\ 0 & 0 & 0 & x+1 \end{vmatrix} \\ &= (x+1)\begin{vmatrix} x+2 & 0 & 2 \\ 1 & x+1 & 2 \\ -1 & 0 & x-1 \end{vmatrix} \\ &= (x+1)^2\begin{vmatrix} x+2 & 2 \\ -1 & x-1 \end{vmatrix} \\ &= (x+1)^2(x^2+x) \\ &= x(x+1)^3 \end{aligned}$$

The zeros of p are -1 and 0, which are the eigenvalues of B.

(b) $$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 1 & 0 & 2 & -1 \\ -1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2-R_1} \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 + 2x_3 - x_4 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} r-2q \\ p \\ q \\ r \end{pmatrix} = p\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + q\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + r\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \text{ for some } p, q, r \in \mathbb{R}.$$

Basis for E_{-1}: $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(c) $\begin{pmatrix} 2 & 0 & 2 & -1 \\ 1 & 1 & 2 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2+R_3]{R_1+2R_3-R_4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3-R_4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\begin{cases} x_2 + x_3 = 0 \\ -(x_1 + x_3) = 0 \\ x_4 = 0 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s \\ -s \\ s \\ 0 \end{pmatrix} = s \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \text{ for some } s \in \mathbb{R}.$$

Basis for E_0: $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

(d) $P = \begin{pmatrix} 0 & -2 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ $D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (Answer is not unique.)

(e) $B^{1101} = (PDP^{-1})^{1101} = PD^{1101}P^{-1} = PDP^{-1} = B$

$$= \begin{pmatrix} -2 & 0 & -2 & 1 \\ -1 & -1 & -2 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Question 5:

(a) $\forall \boldsymbol{v} \in \text{nullspace}(C), C\boldsymbol{v} = \mathbf{0}$

$\implies C^2\boldsymbol{v} = C(C\boldsymbol{v}) = C\mathbf{0} = \mathbf{0}$

$\implies \boldsymbol{v} \in \text{nullspace}(C^2)$

Therefore, $\text{nullspace}(C) \subseteq \text{nullspace}(C^2)$.

(b) From the rank-nullity theorem, $\text{nullity}(C^2) = \text{nullity}(C)$. Since $\text{nullspace}(C) \subseteq \text{nullspace}(C^2)$ (from part a), by Theorem 3.6.9, $\text{nullspace}(C) = \text{nullspace}(C^2)$.

(c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(e) No. $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\} \leq \text{rank}(A)$

Therefore, $\text{rank}(C^2) \leq \text{rank}(C)$.

Question 6:

(a) $$\begin{aligned} T_\lambda(\boldsymbol{u}) &= A\boldsymbol{u} - \lambda\boldsymbol{u} \\ &= A\boldsymbol{u} - \lambda I_n\boldsymbol{u} \\ &= (A - \lambda I_n)\boldsymbol{u} \end{aligned}$$

Standard matrix for $T_\lambda : (A - \lambda I_n)$

(b) $$\begin{aligned} (A - \lambda I)(A - \mu I) &= A^2 - \lambda IA - \mu AI + \lambda\mu I^2 \\ &= A^2 - \lambda AI - \mu IA + \mu\lambda I^2 \\ &= (A - \mu I)(A - \lambda I) \end{aligned}$$

(c) Suppose the eigenvalues of A are $\lambda_1 \dots \lambda_k$.

(i) Since $A\boldsymbol{v} = \lambda_i\boldsymbol{v} = \lambda_i I\boldsymbol{v}$,
$(A - \lambda_i I)\boldsymbol{v} = A\boldsymbol{v} - \lambda_i I\boldsymbol{v} = \boldsymbol{0}$.
Applying the result in part (b) repeatedly yields
$(A - \lambda_1 I)\dots(A - \lambda_i I)\dots(A - \lambda_k I) = (A - \lambda_1 I)\dots(A - \lambda_k I)(A - \lambda_i I)$
Therefore,
$$\begin{aligned} (A - \lambda_1 I)\dots(A - \lambda_i I)\dots(A - \lambda_k I)\boldsymbol{v} &= (A - \lambda_1 I)\dots(A - \lambda_k I)(A - \lambda_i I)\boldsymbol{v} \\ &= (A - \lambda_1 I)\dots(A - \lambda_k I)(\boldsymbol{0}) = \boldsymbol{0} \end{aligned}$$

(ii) Since A is diagonalizable, by Theorem 6.2.3, A has n linearly independent eigenvectors, which will span $\mathbb{R}^n$. Let $\{\boldsymbol{v_1}, \dots, \boldsymbol{v_n}\}$ be one such basis.

$\forall\boldsymbol{v} \in \mathbb{R}^n, \exists a_1, \dots, a_n \in \mathbb{R}$, such that $\boldsymbol{v} = a_1\boldsymbol{v_1} + \dots + a_n\boldsymbol{v_n} = \sum_{k=1}^{n} a_k\boldsymbol{v_k}$.

$$\begin{aligned} S(\boldsymbol{v}) &= S\left(\sum_{k=1}^{n} a_k\boldsymbol{v_k}\right) \\ &= \sum_{k=1}^{n} a_k S(\boldsymbol{v_k}) \\ &= \sum_{k=1}^{n} a_k\boldsymbol{0} \qquad \text{from part (i)} \\ &= \boldsymbol{0} \end{aligned}$$

$\mathrm{R}(S) = \{\boldsymbol{0}\}$, and therefore, S is the zero transformation.