National University of Singapore
Department of Mathematics

Semester 2, 2017/18

MA1101R Linear Algebra I

May 2018 — Time allowed: 2 hours

Student Number: | | | | | | | | | |

Instructions to candidates:

1. This examination paper consists of 6 questions, for a total of 80 points. Excluding the cover page, there are 12 printed pages.
2. Answer all 6 questions.
3. This is a closed book examination, but you are allowed to bring one A4-sized double-sided helpsheet.
4. You are permitted to use any kind of calculator, except devices which can be used for communication and/or web-surfing. However various steps in the calculations should be laid out systematically.
5. Write down your student number in the space provided above. Do not write your name.
6. Write your answers in the space below each question. Only this booklet will be collected at the end of the examination.
7. The blank pages on the left can be used for rough work.

Do not write below this box.

Question:	1	2	3	4	5	6	Total
Points:	10	15	10	15	15	15	80
Score:							

1. Suppose $\mathbf{v}_1 = (1, 1, 1, 1)$, $\mathbf{v}_2 = (1, 2, 4, 5)$, and $\mathbf{v}_3 = (10, -30, -40, -20)$. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and let $U = \text{span}(S)$.

   (a) (6 points) Use the Gram-Schmidt process to find an orthonormal basis T for U.

   (b) (4 points) Find the transition matrix from S to T. You may leave the entries of your answer in terms of square roots.

2. Let $\mathbf{A}$ be the matrix $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

(a) (3 points) Find the characteristic polynomial of $\mathbf{A}$ and verify that the eigenvalues of $\mathbf{A}$ are $\lambda = -1$ and $\lambda = 8$.

(b) (3 points) Find a basis for the eigenspace E_{-1} of $\mathbf{A}$.

(c) (3 points) Find a basis for the eigenspace E_8 of $\mathbf{A}$.

(d) (2 points) Find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.

(e) (4 points) Find a matrix $\mathbf{B}$ such that $\mathbf{B}^3 = \mathbf{A}$. *Show your steps.*

3. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 4 \\ 4 & 10 & 1 \\ 7 & 17 & 3 \\ 2 & 4 & 3 \end{pmatrix}$.

(a) (4 points) Find the rank of $\mathbf{A}$.

(b) (6 points) Find the rank of $\mathbf{B} = \begin{pmatrix} 1 & 3 & 1 & 4 \\ 4 & \lambda & 10 & 1 \\ 7 & 1 & 17 & 3 \\ 2 & 2 & 4 & 3 \end{pmatrix}$. Your answer *will depend* on the value of λ.

4. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(\mathbf{x}) = \left(\frac{\mathbf{u} \cdot \mathbf{x}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u}.$$

(a) (3 points) Verify that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and for any $c, d \in \mathbb{R}$

$$T(c\mathbf{x} + d\mathbf{y}) = cT(\mathbf{x}) + dT(\mathbf{y}),$$

thus proving that T is a linear transformation.

(b) (3 points) Find the standard matrix of the linear transformation T.

(c) (3 points) Find a basis for the kernel of T, and write down the nullity of T.

(d) (3 points) Find a basis for the range of T, and write down the rank of T.

(e) (3 points) Let $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$. It is known that B is a basis for $\mathbb{R}^3$. For any $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, find $[T(\mathbf{x})]_B$.

5. In the question, all vectors are column vectors. Let $\mathbf{A}$ be an $n \times n$ matrix.

   (a) (4 points) Show that for any $\mathbf{u}, \mathbf{w} \in \mathbb{R}^n$, $(\mathbf{Au}) \cdot \mathbf{w} = \mathbf{u} \cdot \left(\mathbf{A}^T\mathbf{w}\right)$.

   (b) (3 points) Give an example of a 2×2 matrix $\mathbf{A}$ for which

$$\left(\mathbf{A}\begin{pmatrix}1\\1\end{pmatrix}\right) \cdot \begin{pmatrix}1\\0\end{pmatrix} \neq \begin{pmatrix}1\\1\end{pmatrix} \cdot \left(\mathbf{A}\begin{pmatrix}1\\0\end{pmatrix}\right).$$

   You should demonstrate that your example works.

(c) (3 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be orthonormal vectors in $\mathbb{R}^n$. Suppose $\mathbf{w} \in \mathbb{R}^n$.
Define $b_1 = (\mathbf{A}\mathbf{v}_1) \cdot \mathbf{w}$, $b_2 = (\mathbf{A}\mathbf{v}_2) \cdot \mathbf{w}$, and $b_3 = (\mathbf{A}\mathbf{v}_3) \cdot \mathbf{w}$.
Define $\mathbf{q} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3$.
Calculate $\mathbf{v}_1 \cdot \mathbf{q}$, $\mathbf{v}_2 \cdot \mathbf{q}$, and $\mathbf{v}_3 \cdot \mathbf{q}$ (hint: recall $\mathbf{v}_1 \cdot \mathbf{v}_1 = 1$, $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ etc.).

(d) (5 points) Using the same definitions as in Part(c), show that for every $\mathbf{v} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $(\mathbf{A}\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{q}$.

6. Let $\mathbf{A}$ be an $n \times n$ matrix.

   (a) (3 points) Suppose λ is an eigenvalue of $\mathbf{A}$ and suppose $\mathbf{v}$ is an eigenvector associated with λ. Show that for any $m > 0$, $\mathbf{A}^m\mathbf{v} = \lambda^m\mathbf{v}$.

   Hint: use induction.

   (b) (2 points) Let λ be a real number which is an eigenvalue of $\mathbf{A}$. Show that if $m > 0$ and $\mathbf{A}^m = \mathbf{I}$, then $\lambda = \pm 1$.

(c) (2 points) Suppose $\mathbf{P}$ and $\mathbf{B}$ are $n \times n$ matrices such that $\mathbf{P}$ is invertible and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{B}$. Show that if $\mathbf{B}^2 = \mathbf{I}$, then $\mathbf{A}^2 = \mathbf{I}$.

(d) (8 points) Assume that $\mathbf{A}$ is a symmetric matrix. Show that if $m > 0$ and $\mathbf{A}^m = \mathbf{I}$, then $\mathbf{A}^2 = \mathbf{I}$.