NATIONAL UNIVERSITY OF SINGAPORE

MA1101R - LINEAR ALGEBRA I

(Semester 1 : AY2018/2019)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Please write your matriculation/student number only. Do not write your name.

2. This examination paper contains **SIX** questions and comprises **FOUR** printed pages.

3. Answer **ALL** questions.

4. Please start each question on a new page.

5. This is a CLOSED BOOK (with helpsheet) examination.

6. Students are allowed to use one A4 size helpsheet.

7. Candidates may use scientific calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [16 marks]

(a) Consider the following matrix

$$\boldsymbol{C} = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 5 & -3 & 2 \\ 2 & 4 & -2 & 0 \\ 3 & 8 & -5 & 4 \end{pmatrix}.$$

(i) Find a basis for the row space of $\boldsymbol{C}$.

(ii) Find a basis for the nullspace of $\boldsymbol{C}$. What is the rank and nullity of $\boldsymbol{C}$?

(iii) Is the last row of $\boldsymbol{C}$ a linear combination of the other rows of $\boldsymbol{C}$? If it is, find such a linear combination. If it is not, explain why.

(b) Suppose $\boldsymbol{D}$ is a matrix with k columns such that the linear system $\boldsymbol{Dx} = \boldsymbol{r}$ is consistent for all vectors $\boldsymbol{r} \in \mathbb{R}^n$. For each of the statements below, determine if the statement is true. Justify your answer.

(i) $\boldsymbol{D}$ has n rows.

(ii) k is at least n.

(iii) $\boldsymbol{D}$ is of full rank.

Question 2 [20 marks]

(a) $\boldsymbol{A}$ is a square matrix of order 10 with entries a_{ij} such that $\det(\boldsymbol{A}) = 2$. Let $\boldsymbol{B}$ be another square matrix of order 10 such that

$$b_{ij} = \begin{cases} -\frac{1}{2}a_{ij} & \text{if } i \text{ is odd;} \\ 2a_{ij} & \text{if } i \text{ is even.} \end{cases}$$

Find $\det(\boldsymbol{B})$.

(b) Let $\boldsymbol{B} = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 10 & 5 \\ 1 & -2 & 3 \end{pmatrix}$.

Perform **three** elementary row operations to reduce $\boldsymbol{B}$ to a row-echelon form. Hence find three elementary matrices $\boldsymbol{E_1}$, $\boldsymbol{E_2}$, $\boldsymbol{E_3}$ such that $\boldsymbol{B}^T\boldsymbol{E_1}\boldsymbol{E_2}\boldsymbol{E_3}$ is a <u>lower triangular</u> matrix. Write down the elementary row operations that $\boldsymbol{E_1}, \boldsymbol{E_2}, \boldsymbol{E_3}$ represents respectively.

(c) Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ be square matrices of the same order. Prove the following statements.

(i) $\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{0}$ if and only if $\boldsymbol{X} = \boldsymbol{0}$. (Hint: Consider the diagonal entries of $\boldsymbol{X}^T\boldsymbol{X}$.)

(ii) $\boldsymbol{XY} = \boldsymbol{0}$ if and only if $\boldsymbol{X}^T\boldsymbol{XY} = \boldsymbol{0}$. (Hint: Use part (i).)

Question 3 [16 marks]

(a) Let $V = \{(a-b, a-2b, a+b, a+3b) \mid a, b \in \mathbb{R}\}$.

(i) Show that V is a subspace of $\mathbb{R}^4$.

(ii) Find a basis for V. What is the dimension of V?

(iii) Let W be the solution space of the following homogeneous linear system:

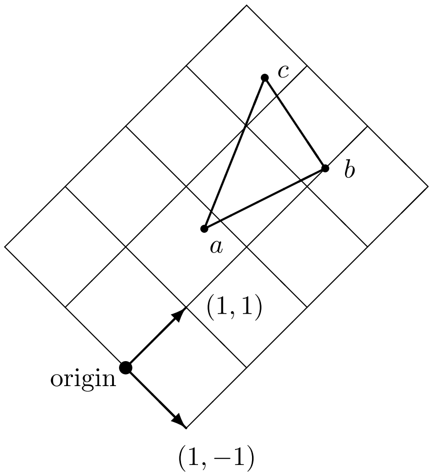
$$\begin{cases} x_1 - x_2 + x_3 + x_4 = 0 \\ \qquad x_2 - x_3 + 6x_4 = 0 \\ \qquad\qquad x_3 + 3x_4 = 0 \end{cases}$$

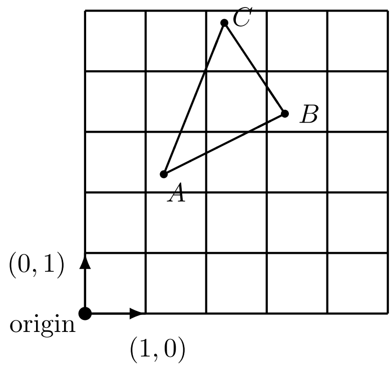
Find a basis for W and hence show that $W \subseteq V$.

(b) Let $E = \{(1,0),(0,1)\}$ and $S = \{(1,1),(1,-1)\}$.

(i) E is the standard basis for $\mathbb{R}^2$. Is S also a basis for $\mathbb{R}^2$? Justify your answer.

(ii) The triangle in the right figure is re-drawn exactly on the left figure as shown. Find the coordinates of the 3 points a, b and c, with respect to the coordinates used in the left figure. The coordinates of A, B and C are given by $(1.2, 2.2)$, $(3.2, 3.2)$ and $(2.2, 4.8)$ respectively.





Question 4 [20 marks]

Let $V = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$, where

$$\boldsymbol{u_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{u_3} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}.$$

(i) Find a vector $\boldsymbol{u}$ such that $||\boldsymbol{u}|| = 3\sqrt{10}$ and $\boldsymbol{u}$ is orthogonal to $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$.

(ii) Use the Gram-Schmidt Process to transform $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ into an orthonormal basis for V.

(iii) Find the projection of $\boldsymbol{w} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 13 \end{pmatrix}$ onto V.

(iv) Let $\boldsymbol{A} = (\boldsymbol{u_1}\ \boldsymbol{u_2}\ \boldsymbol{u_3})$, that is, $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ are the columns of $\boldsymbol{A}$. Find a least squares solution to the linear system $\boldsymbol{Ax} = \boldsymbol{w}$.

Question 5 [20 marks]

(a) Let $\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 3 \\ 1 & 1 & -1 \end{pmatrix}$.

(i) Find all the eigenvalues of $\boldsymbol{A}$.

(ii) Find a basis for the eigenspace associated with each eigenvalue of $\boldsymbol{A}$.

(iii) Find a matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$ and determine $\boldsymbol{P}^{-1}\boldsymbol{AP}$.

(b) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of order n. Suppose $\boldsymbol{AB} = \boldsymbol{BA}$ and $\boldsymbol{A}$ has n distinct eigenvalues.

(i) Show that each eigenspace of $\boldsymbol{A}$ has dimension 1.

(ii) Show that if $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$, then $\boldsymbol{u}$ is also an eigenvector of $\boldsymbol{B}$.

(iii) Show that $\boldsymbol{A}$ and $\boldsymbol{B}$ are simultaneously diagonalizable, i.e., there exists an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{PAP}^{-1}$ and $\boldsymbol{PBP}^{-1}$ are diagonal.

Question 6 [8 marks]

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation such that

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x - y + z \\ 2x - y - z \end{pmatrix} \quad \text{for all } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

Find 2 **different** linear transformations S_1 and S_2 such that $(T \circ S_1)$ and $(T \circ S_2)$ are both the identity linear operator on $\mathbb{R}^2$, **showing clearly** how S_1 and S_2 are derived. Give your answers by providing the formulae for S_1 and S_2.

End of Paper