National University of Singapore
Department of Mathematics

Semester 2, 2018/19

MA1101R Linear Algebra I

May 2019 — Time allowed: 2 hours

Student Number:

Instructions to candidates:

1. This examination paper consists of 6 questions, for a total of 80 points. Excluding the cover page, there are 12 printed pages.
2. Answer all 6 questions.
3. This is a closed book examination, but you are allowed to bring one A4-sized double-sided helpsheet.
4. You are permitted to use any kind of calculator, except devices which can be used for communication and/or web-surfing. However various steps in the calculations should be laid out systematically.
5. Write down your student number in the space provided above. Do not write your name.
6. Write your answers in the space below each question. Only this booklet will be collected at the end of the examination.
7. The blank pages on the left can be used for rough work.

Do not write below this box.

Question:	1	2	3	4	5	6	Total
Points:	15	10	15	10	15	15	80
Score:							

1. Let $\mathbf{v}_1 = (4, 0, 2, 4)$, $\mathbf{v}_2 = (3, 0, 3, 0)$, and $\mathbf{v}_3 = (1, 4, 3, 5)$. Let $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

   (a) (4 points) Apply the **Gram-Schmidt Process** to transform $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ into an **orthonormal** basis for V.

   (b) (4 points) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and let T be the orthonormal basis for V found in Part (a). Find the transition matrix from S to T.

(c) (3 points) Let $\mathbf{u} = (0, -5, 16, 13)$. Find the orthogonal projection of $\mathbf{u}$ onto V.

(d) (4 points) Let $\mathbf{A} = \begin{pmatrix} 4 & 3 & 1 \\ 0 & 0 & 4 \\ 2 & 3 & 3 \\ 4 & 0 & 5 \end{pmatrix}$. Find a **least squares solution** to $\mathbf{Ax} = \mathbf{u}$.

2. Let a, b, c be real numbers. Assume that a, b, c are **not all equal.** This means we **cannot have** $a = b = c$. Let

$$\mathbf{A} = \begin{pmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{pmatrix}.$$

(a) (5 points) Find the rank of $\mathbf{A}$.

(b) (5 points) Find a basis for the nullspace of $\mathbf{A}$.

3. Let $\mathbf{A}$ be the matrix $\begin{pmatrix} 7 & -6 & 2 \\ 16 & -13 & 4 \\ 24 & -18 & 5 \end{pmatrix}$.

(a) (3 points) Find the characteristic polynomial of $\mathbf{A}$ and verify that the eigenvalues of $\mathbf{A}$ are $\lambda = 1$ and $\lambda = -1$.

(b) (3 points) Find a basis for the eigenspace E_1 of $\mathbf{A}$ corresponding to the eigenvalue $\lambda = 1$.

(c) (3 points) Find a basis for the eigenspace E_{-1} of $\mathbf{A}$ corresponding to the eigenvalue $\lambda = -1$.

(d) (2 points) Find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.

(e) (4 points) Calculate $\mathbf{A}^{2019}$. **You must show your steps.**

4. Consider the subspace V of $\mathbb{R}^4$ defined by $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : x_1 - x_2 + 2x_3 + 6x_4 = 0 \right\}$.

(a) (5 points) Find a basis for V.

(b) (5 points) Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that the kernel of T is $\{\mathbf{0}\}$ and the range of T is V. Write down the standard matrix of T.

5. Let $\mathbf{B}$ be any $m \times m$ matrix. Suppose $\mathbf{P}$ is an invertible $m \times m$ matrix and that $\mathbf{C} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$.

   (a) (6 points) Show that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for the nullspace of $\mathbf{C}$, then $\{\mathbf{P}\mathbf{v}_1, \ldots, \mathbf{P}\mathbf{v}_k\}$ is a basis for the nullspace of $\mathbf{B}$.

(b) (3 points) Prove that rank(**B**) = rank(**C**).

(c) (6 points) Let **A** be any $m \times p$ matrix. Suppose that the linear system $\mathbf{Ax} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$. Prove that the linear system $\mathbf{A}^T\mathbf{y} = \mathbf{0}$ has only the trivial solution.

(The matrix **A** is not related to **B** or **C** above.)

6. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices.

   (a) (4 points) Show that if there is an invertible $n \times n$ matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ and $\mathbf{P}^{-1}\mathbf{BP}$ are both diagonal matrices, then $\mathbf{AB} = \mathbf{BA}$.

   (b) (5 points) Assume that $\mathbf{A}$ has n distinct eigenvalues. Show that if every eigenvector of $\mathbf{A}$ is an eigenvector of $\mathbf{B}$ (possibly associated with a different eigenvalue), then $\mathbf{AB} = \mathbf{BA}$.

(c) (6 points) Assume again that $\mathbf{A}$ has n distinct eigenvalues. Show that if $\mathbf{AB} = \mathbf{BA}$, then every eigenvector of $\mathbf{A}$ is an eigenvector of $\mathbf{B}$ (possibly associated with a different eigenvalue).

(Remember that an eigenvector is non-zero by definition.)