NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2014–2015

MA1102R Calculus

April 28, 2014 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **EIGHT (8)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. Write your solutions in the **ANSWER BOOK**.
3. Please start each question on a new page.
4. Please write your student number only. Do **NOT** write your name on the answer book.
5. Total marks for this exam are **60**. The marks for each question are indicated at the beginning of each question.
6. This is a **CLOSED BOOK** examination. **Two A4-sized helpsheet (two-sided) are allowed.**
7. Candidates may use **scientific calculators**. However, they should lay out systematically the various steps in the calculations.

Question 1 [10 marks]

Evaluate the following.

(1) $\lim_{x\to 1}(\frac{x}{x-1} - \frac{1}{\ln x})$.

(2) $\frac{d}{dx}x^{x\ln x}$.

Question 2 [7 marks]

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that

$$\lim_{x\to -\infty} f(x) = \lim_{x\to\infty} f(x) = 0.$$

By making use of the extreme value theorem or otherwise, prove that $f(x)$ is bounded on $\mathbb{R}$.

Question 3 [7 marks]

Let $g : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$g(x) = \begin{cases} x^2 \sin(1/x^2) & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0. \end{cases}$$

Show that $g(x)$ is differentiable for all $x \in \mathbb{R}$ and calculate $g'(x)$.

Question 4 [9 marks]

The Young's inequality asserts: let a, b, p, q be positive real number. Assume further that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab.$$

Answer the following questions.

(1) Define

$$f(t) = \frac{t^q}{q} - t + \frac{1}{p}$$

for $t \geq 0$. Prove that $f(t) \geq 0$ for $t \geq 0$.

(2) Hence or otherwise, prove the Young's inequality.
(Hint: consider $t = b/a^{p-1}$.)

Question 5 [6 marks]

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Show that if

$$\lim_{x \to a} f'(x) = A,$$

then

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = A.$$

Question 6 [6 marks]

Prove the following integral mean value theorem. Let $f(x)$ be a continuous function on $[a, b]$. Show that there is a $c \in (a, b)$ such that

$$\frac{\int_a^b f(x)dx}{b - a} = f(c).$$

Question 7 [7 marks]

For $a > 0$ is a fixed constant, let C be the loop defined by the equation

$$3ay^2 = x(a - x)^2.$$

Find the area of the surface generated by rotating the loop C about the x-axis.

Question 8 [8 marks]

For each non-negative integer n, define the Legendre polynomial

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}((x^2 - 1)^n).$$

Use integration by parts or otherwise, prove that if $m \neq n$, then

$$\int_{-1}^{1} P_m(x) P_n(x) dx = 0.$$

End of paper.