NATIONAL UNIVERSITY OF SINGAPORE

MA1102R — CALCULUS

(Semester 2 : AY2015/2016)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Write your matriculation/student number only. Do not write your name.
2. This examination paper contains a total of **SIX (6)** questions and comprises **FOUR (4)** printed pages.
3. This examination carries a total of **100** marks.
4. Answer **ALL** questions.
5. This is a **CLOSED BOOK** examination.
6. Your are allowed to use four A4-sized, double-sided help sheets.
7. You may use non-graphing and non-programmable calculators. However, you should lay out systematically the various steps in the calculations.

Question 1 **[15 marks]**

(a) Solve the differential equation

$$(x^2 - 4)\,\frac{dy}{dx} + 2xy = 4y + 12 \qquad (x > 2).$$

(b) Show that the substitution $y = zx^2$ reduces the differential equation

$$x^3 \frac{dy}{dx} = 2y^2 + 4x^2 y \qquad (x > 0)$$

to

$$x\,\frac{dz}{dx} = f(z),$$

where the function $f(z)$ is to be determined.

Given that $y = 1$ when $x = 1$, express y in terms of x.

Question 2 **[15 marks]**

The function f is defined on $[0, 1]$ by

$$f(x) = e^{2\sqrt{x}}.$$

(i) Show that

$$f''(x) = \frac{ke^{2\sqrt{x}}\,(2\sqrt{x} - 1)}{x\sqrt{x}},$$

where k is a constant to be determined.

(ii) Find the interval(s) on which f is concave upward.

(iii) Sketch the graph of f.

(iv) Use the substitution $x = u^2$ to find the exact area of the region enclosed by the graph of f , the axes and the line $x = 1$.

(v) Use your answer in (iv) to deduce the exact value of

$$\int_1^{e^2} (\ln y)^2\, dy.$$

Question 3 **[15 marks]**

(a) For $n \geq 0$, let

$$I_n = \int_0^8 x^n (8-x)^{1/3}\,dx.$$

(i) Find the value of I_0.

(ii) Show that for $n \geq 1$,

$$I_n = \frac{24n}{3n+4}\, I_{n-1}.$$

(iii) R is the region bounded by the axes and the curve $y = (8-x)^{4/3}$ for $0 \leq x \leq 8$. Use the answer from (i) and the reduction formula in (ii) to find the volume of the solid formed by rotating R completely about the y-axis.

(b) A curve C has equation

$$y = \sec(2x), \qquad 0 \leq x \leq \frac{\pi}{6}.$$

Find the exact length of the curve C.

Question 4 **[15 marks]**

Let $P(p^2, 2p)$, where $p > 0$, be a variable point on the curve whose equation is $y^2 = 4x$. The normal at P meets the curve again at the point $Q(q^2, 2q)$.

(i) Express q in terms of p.

(ii) Prove that the length, L, of the line segment PQ is given by

$$L^2 = \frac{k(1+p^2)^3}{p^4},$$

where k is a constant to be determined.

(iii) Find the value of p for which L attains its least value. Justify your answer.

Question 5 **[20 marks]**

(a) Use Riemann sum to find
$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{2n^3+k^3}.$$

(b) Find the exact value of
$$\lim_{x\to 0}\frac{\displaystyle\int_{e^x}^{e^{5x}}\left(e^{x+t^2}\right)dt}{\ln(3x+1)}.$$

(c) Let $[x]$ denote the greatest integer not exceeding x. Find the exact value of
$$\int_1^{2016}\frac{[\ln x]}{x}\,dx.$$

(d) Let f be continuous on $[-a,a]$, where a is a positive constant. Show that
$$\int_{-a}^{a}f(x)\,dx=\int_0^a\Big(f(x)+f(-x)\Big)\,dx.$$
Hence, find the exact value of
$$\int_{-1}^{1}\frac{x^{2016}e^x}{e^x+1}\,dx.$$

Question 6 **[20 marks]**

(a) Use the Rolle's Theorem to show that the equation
$$(x^2-6x+12)e^x=2015x+2016$$
has at most two real roots.

(b) Find the exact value of
$$\lim_{x\to 0}\left(3-2\cos 3x+3\sin 4x\right)^{4\cot 2x}.$$

(c) Let a be a positive real number. Use the precise definition of limit to prove that
$$\lim_{x\to a}\frac{3x^2+2ax+a^2}{2x+a}=2a.$$

(d) Let f be continuous on $[0,1]$ and differentiable on $(0,1)$ such that $f(0)=1$ and $f(1)=0$ and let k be a positive constant.

(i) Show that there exists c in $(0,1)$ such that $f(c)=kc$.

(ii) Show that there exist a and b in $(0,1)$ such that
$$cf'(a)+(1-c)f'(b)=-1.$$

END OF PAPER