NATIONAL UNIVERSITY OF SINGAPORE

MA1102R — CALCULUS

2017 – 2018 SEMESTER 1

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. Do not write your name.
2. This examination paper contains a total of **EIGHT (8)** questions and comprises **FOUR (4)** printed pages.
3. Answer **ALL** questions. The marks for questions are not necessarily the same; marks for each question are indicated at the beginning of the question.
4. Use a separate page for each question.
5. This is a **CLOSED BOOK** examination.
6. Two pieces of A4-size formula sheet are prepared and provided by the examiners.
7. Candidates may use non-graphing and non-programmable calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [10 marks]

Let $f(x) = (x^3 + 4x^2 + 11x + 14)e^{-x}$.

(i) Find the open intervals on which f is increasing and decreasing.

(ii) Find the local maximum and minimum values of f.

(iii) Find the open intervals on which f is concave up and concave down.

(iv) Find the coordinates of all the inflection points of f.

Question 2 [17 marks]

(a) Using only the ϵ, δ-definition, prove that

$$\lim_{x \to 2} \frac{x}{x^2 + 2} = \frac{1}{3}.$$

(b) Express the following limit as a definite integral:

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^3}{n^2(n^2 + i^2)}.$$

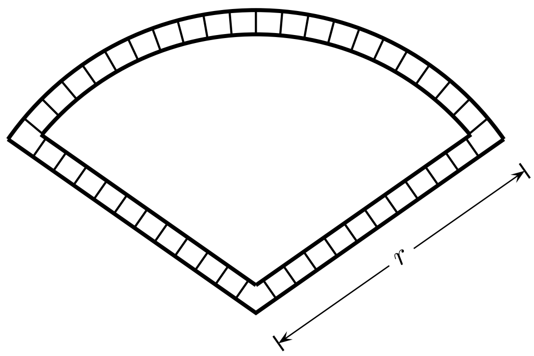
Hence, evaluate the limit.

(c) Evaluate the limit

$$\lim_{x \to 0^+} \left(\frac{e^x - 1}{x} \right)^{1/x}.$$

Question 3 [8 marks]

A gardener has 50 m of interlocking stone available for fencing off a flower bed in the form of a circular sector. Find the radius of the circle that will yield a flower bed with the largest area if the gardener uses all of the stone. Justify your answer.



Question 4 [19 marks]

(a) Find $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{4}$ if

$$y = (\tan x)^{\sec x}(\sec x)^{\tan x}.$$

(b) Let $F(x) = \displaystyle\int_0^{x^2} f(t)\,dt$, where

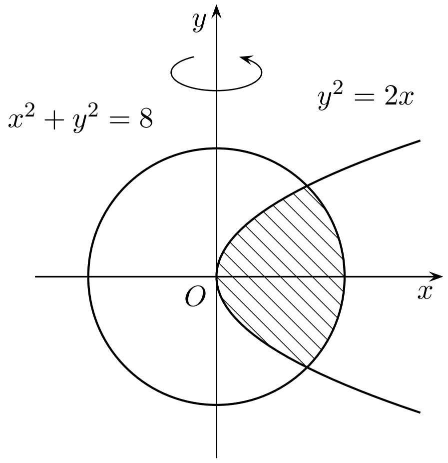
$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Find all the critical numbers of F, and classify whether F attains a local maximum or minimum value at each of these points.

(c) Let f be a function twice differentiable on $\mathbb{R}$. Suppose that $f(0) > 0$, $f'(0) = 0$ and $f''(x) < 0$ for all $x > 0$. Prove that the equation $f(x) = 0$ has exactly one positive root.

Question 5 [12 marks]

Let R be the region bounded between the circle $x^2 + y^2 = 8$ and the parabola $y^2 = 2x$.



(i) Find the area of R.

(ii) If R is rotated about the y-axis, find the volume of the resulting solid.

Question 6 [13 marks]

(i) Find $\displaystyle\int \frac{x \ln x}{(1 + x^2)^2}\,dx$.

(ii) Evaluate $\displaystyle\int_0^{\infty} \frac{x \ln x}{(1 + x^2)^2}\,dx$.

Question 7 [15 marks]

(a) Consider the initial value problem

$$\frac{dy}{dx} = y^2 - \frac{y}{x} - \frac{1}{x^2} \quad (x > 0), \quad y = 2 \text{ at } x = 1.$$

Using the substitution $y = \frac{1}{x} + \frac{1}{z}$, convert the differential equation into

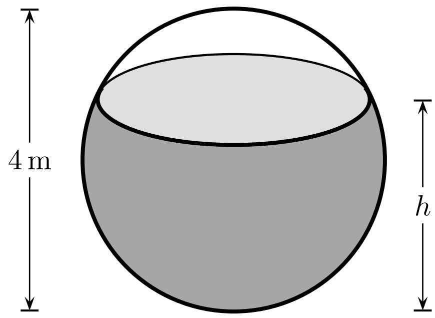
$$\frac{dz}{dx} + \frac{z}{x} + 1 = 0.$$

Hence, solve the given initial value problem.

(b) A spherical container of radius 2 m is initially full of water. Suppose that water drains through a small hole at the bottom of the container. Let t be the time (in minute), and h be the depth of the water (in metre) at time t. It is known that t and h satisfy the following differential equation:

$$(4h - h^2)\frac{dh}{dt} = -\sqrt{h}.$$

How long will it take for the water to drain completely?



Question 8 [6 marks]

Let f be a differentiable function such that f' is continuous on $[0,1]$, and M the maximum value of $|f'(x)|$ on $[0,1]$. Prove that if $f(0) = f(1) = 0$, then

$$\int_0^1 |f(x)|\,dx \leq \frac{M}{4}.$$

END OF PAPER