Question 1:

(i) f is continuous and differentiable everywhere. The first derivative

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^3+4x^2+11x+14)e^{-x} + (x^3+4x^2+11x+14)\frac{d}{dx}(e^{-x}) \\ &= (3x^2+8x+11)e^{-x} + (x^3+4x^2+11x+14)(-e^{-x}) \\ &= (-x^3-x^2-3x-3)e^{-x} \\ &= -(x+1)(x^2+3)e^{-x} \end{aligned}$$

is zero at $x=-1$.

When $x<-1$, $f'(x)>0$. Applying the Increasing Test, f is increasing on $(-\infty,-1)$.

When $x>-1$, $f'(x)<0$. Applying the Decreasing Test, f is decreasing on $(-1,\infty)$.

(ii) $x=-1$ is the only critical point.

Since $f'(x)$ changes from positive to negative, from the First Derivative Test, f has a local maximum at $x=-1$. The value of the local maximum is $f(-1)=6e$.

(iii) The second derivative

$$\begin{aligned} f''(x) &= \frac{d}{dx}(-x^3-x^2-3x-3)e^{-x} + (-x^3-x^2-3x-3)\frac{d}{dx}(e^{-x}) \\ &= (-3x^2-2x-3)e^{-x} + (-x^3-x^2-3x-3)(-e^{-x}) \\ &= (x^3-2x^2+x)e^{-x} \\ &= x(x-1)^2e^{-x} \end{aligned}$$

is zero when $x=0$ or $x=1$.

When $0<x<1$, $f''(x)>0$. f' is increasing on $[0,1]$.

When $1<x$, $f''(x)>0$. f' is increasing on $[1,\infty)$.

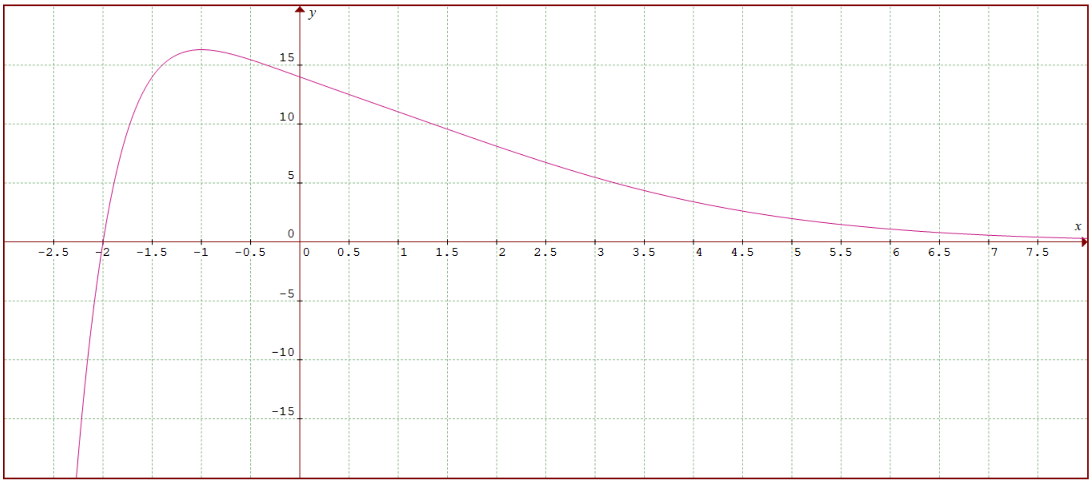
Since f' is increasing on $[0,\infty)$, f is concave up on $(0,\infty)$.

When $x<0$, $f''(x)<0$. Applying the Concavity Test, f is concave down on $(-\infty,0)$.

(iv) Since the concavity only changes at $x=0$, that corresponds to the only inflection point.

$f(0)=14$

Coordinates: $(0,14)$



Graph of $y = f(x)$

Question 2:

(a) Given $\epsilon > 0$, let $\delta = \min\{1, 3\epsilon\}$. If $0 < |x-2| < \delta$, then

$$
\begin{aligned}
\left|\frac{x}{x^2+2} - \frac{1}{3}\right| &= \left|\frac{3x-(x^2+2)}{3(x^2+2)}\right| \\
&= \left|\frac{-(x-2)(x-1)}{3(x^2+2)}\right| \\
&\leq \left|\frac{(x-2)(x-1)}{6}\right| && \left|\tfrac{-1}{x^2+2}\right| \leq \tfrac{1}{2} \\
&= \frac{|(x-2)((x-2)+1)|}{6} \\
&\leq \frac{|x-2|(|x-2|+1)}{6} && \text{triangle inequality} \\
&< \frac{\delta(\delta+1)}{6} \\
&\leq \frac{\delta(2)}{6} && \delta \leq 1 \\
&\leq \epsilon && \delta \leq 3\epsilon
\end{aligned}
$$

(b) Expressing the limit as a Reimann sum

$$
\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^2(n^2+i^2)} &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{(i/n)^3}{1+(i/n)^2} \\
&= \int_0^1 \frac{x^3}{1+x^2} dx
\end{aligned}
$$

Using the substitution $u = 1 + x^2$, $\frac{du}{dx} = 2x$

$$\begin{aligned}\int_0^1 \frac{x^3}{1+x^2}dx &= \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}(2x)dx \\ &= \frac{1}{2}\int_1^2 \frac{u-1}{u}du \\ &= \frac{1}{2}[u - \ln|u|]_1^2 \\ &= \frac{1-\ln 2}{2}\end{aligned}$$

(c) When $x > 0$, both $e^x - 1 > 0$ and $\frac{1}{x} > 0$. Using the fact that $\exp(\ln x) = x$,

$$\begin{aligned}\lim_{x\to 0^+}\left(\frac{e^x-1}{x}\right)^{\frac{1}{x}} &= \lim_{x\to 0^+}\exp\left(\frac{1}{x}\ln\left(\frac{e^x-1}{x}\right)\right) \\ &= \exp\left(\lim_{x\to 0^+}\frac{1}{x}\ln\left(\frac{e^x-1}{x}\right)\right)\end{aligned}$$

Using L'hopital Rule,

$$\begin{aligned}\lim_{x\to 0^+}\frac{1}{x}\ln\left(\frac{e^x-1}{x}\right) &= \lim_{x\to 0^+}\frac{\frac{d}{dx}\left(\ln\left(\frac{e^x-1}{x}\right)\right)}{\frac{d}{dx}(x)} && \tfrac{0}{0}\text{ indeterminate form} \\ &= \lim_{x\to 0^+}\left(\frac{e^x-1}{x}\right)^{-1}\frac{d}{dx}\left(\frac{e^x-1}{x}\right) \\ &= \lim_{x\to 0^+}\frac{x}{e^x-1}\,\frac{(e^x)x-(e^x-1)(1)}{x^2} \\ &= \lim_{x\to 0^+}\frac{xe^x-e^x+1}{x(e^x-1)} \\ &= \lim_{x\to 0^+}\frac{x-1+e^{-x}}{x-xe^{-x}} && \text{still } \tfrac{0}{0} \\ &= \lim_{x\to 0^+}\frac{\frac{d}{dx}(x-1+e^{-x})}{\frac{d}{dx}(x-xe^{-x})} \\ &= \lim_{x\to 0^+}\frac{1-e^{-x}}{1-(e^{-x}-xe^{-x})} && \text{still } \tfrac{0}{0} \\ &= \lim_{x\to 0^+}\frac{\frac{d}{dx}(1-e^{-x})}{\frac{d}{dx}(1-e^{-x}+xe^{-x})} \\ &= \lim_{x\to 0^+}\frac{e^{-x}}{e^{-x}+(e^{-x}-xe^{-x})} \\ &= \frac{e^{-(0)}}{e^{-(0)}+e^{-(0)}-(0)e^{-(0)}} = \frac{1}{2}\end{aligned}$$

$$\lim_{x\to 0^+}\left(\frac{e^x-1}{x}\right)^{\frac{1}{x}} = \exp\left(\frac{1}{2}\right) = \sqrt{e}$$

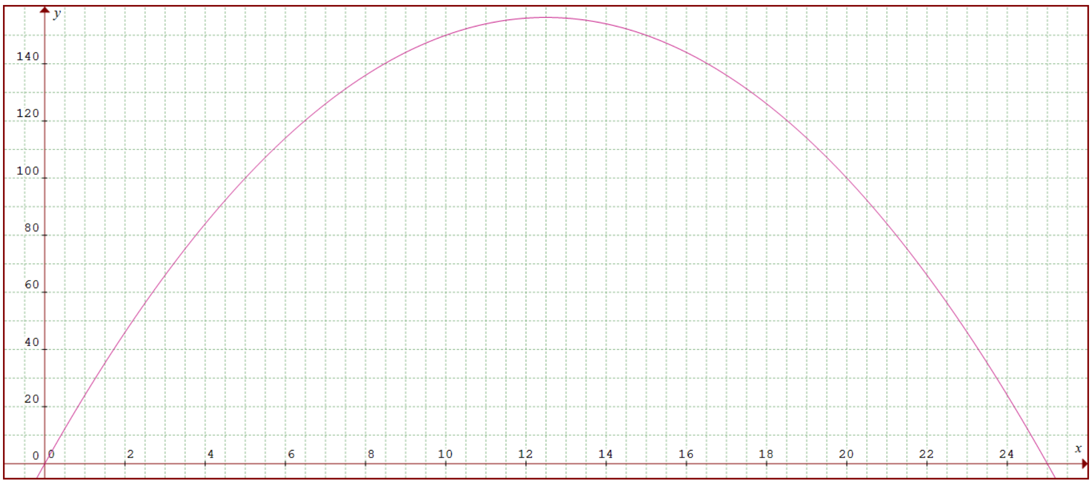
Question 3:
Let θ be the angle (in radians) subtended by the arc.

$$\begin{aligned} \text{Perimeter} &= r + r + r\theta = 50\text{m} \\ \theta &= \frac{50\text{m} - 2r}{r} \\ A &= \frac{1}{2}r^2\theta \\ &= \frac{1}{2}r(50\text{m} - 2r) \end{aligned}$$

To use the Closed Interval Method, solve for the critical numbers.

$$\begin{aligned} \frac{dA}{dr} &= \frac{d}{dr}\left(\frac{1}{2}((50\text{m})r - 2r^2)\right) \\ &= \frac{1}{2}(50\text{m} - 4r) \\ &= 0 \\ r &= 12.5\text{m} \end{aligned}$$

When $r = 12.5$m, $A = 156.25\text{m}^2$. Next, check the values at the endpoints. When $r = 0$m or $r = 50$m, $A = 0\text{m}^2$. Therefore, a radius of 12.5m will yield the largest area of 156.25m^2.



Graph of Area Enclosed against Radius

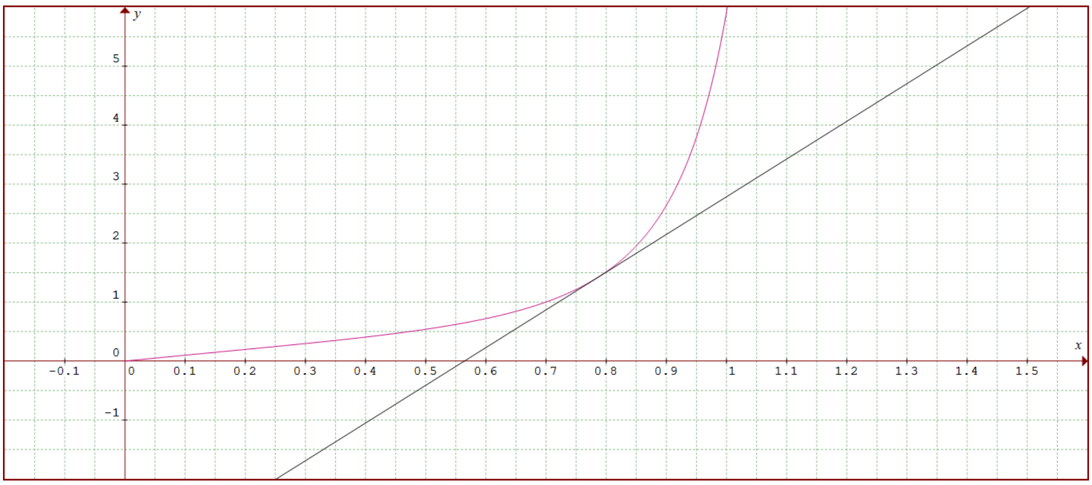
Question 4:

(a) When $x = \frac{\pi}{4}$, $y = (1)^{\sqrt{2}}(\sqrt{2})^1 = \sqrt{2}$. Let $0 < x < \frac{\pi}{2}$. Since the equation is always positive, taking the natural logarithm on both sides gives

$$\ln y = \sec x \ln(\tan x) + \tan x \ln(\sec x)$$

Performing implicit differentiation,

$$\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{d}{dx}(\sec x \ln(\tan x)) + \frac{d}{dx}(\tan x \ln(\sec x)) \\
\frac{1}{y}\frac{dy}{dx} &= \sec x \tan x \ \ln(\tan x) + \sec x \ \frac{\sec^2 x}{\tan x} \\
&\quad + \sec^2 x \ \ln(\sec x) + \tan x \ \frac{\sec x \tan x}{\sec x} \\
&= \sec x \tan x \ln(\tan x) + \frac{\sec^3 x}{\tan x} \\
&\quad + \sec^2 x \ln(\sec x) + \tan^2 x \\
\frac{1}{\sqrt{2}}\frac{dy}{dx}\bigg|_{x=\frac{\pi}{4}} &= \sqrt{2}(1)\ln 1 + \frac{(\sqrt{2})^3}{1} + (\sqrt{2})^2 \ln\sqrt{2} + 1^2 \\
&= 2\sqrt{2} + \ln 2 + 1 \\
\frac{dy}{dx}\bigg|_{x=\frac{\pi}{4}} &= 4 + \sqrt{2}(\ln 2 + 1)
\end{aligned}$$



Graph of $y = (\tan x)^{\sec x}(\sec x)^{\tan x}$ and its tangent line at $x = \frac{\pi}{4}$

(b) The upper limit of integration is not x but x^2. This makes F a composite function.

$$F(u) = \int_0^u f(t)dt \qquad \text{and} \qquad u = x^2$$

To find the critical numbers, the Chain Rule is needed.

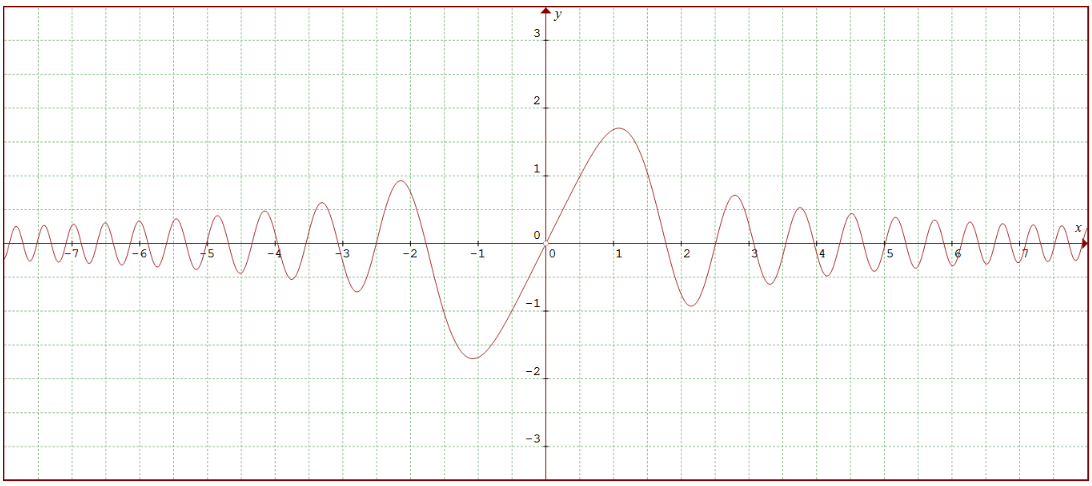
$$\frac{d}{dx}F(x) = \frac{d}{du}(F(u))\frac{du}{dx} = \frac{d}{du}\left(\int_0^u f(t)dt\right)\frac{du}{dx}$$

From the Fundamental Theorem of Calculus, $\frac{d}{du}\int_0^u f(t)dt = f(u)$. Thus, the derivative

$$\begin{aligned}\frac{d}{dx}F(x) &= \frac{du}{dx}f(u)\\ &= 2xf(x^2)\\ &= \begin{cases}\frac{2\sin(x^2)}{x}, & \text{if } x \neq 0,\\ 0, & \text{if } x = 0.\end{cases}\end{aligned}$$

is zero iff $x^2 = k\pi$ for some integer k.

When k is odd, $\frac{\sin(x^2)}{x}$ changes from positive to negative, and F attains a local maximum.
When k is even, $\frac{\sin(x^2)}{x}$ changes from negative to positive, and F attains a local minimum.



Graph of $y = \frac{2\sin(x^2)}{x}$

(c) From the Decreasing Test, $f''(x) < 0$ means that f' is decreasing on $[0, \infty)$. Using the definition of a decreasing function, for all $x > 0$, $f'(x) < f'(0) = 0$. Since $f'(x)$ is negative, f is also decreasing on $[0, \infty)$. Now, suppose f has more than one positive root. There would then exist $0 < x_1 < x_2$, such that $f(x_1) = f(x_2) = 0$. However, since f is decreasing, $f(x_1) > f(x_2)$. A contradiction! Therefore, f has at most 1 positive root.

To show that f has at least one positive root, consider some $a > 0$.
Case $f(a) = 0$: Shown.
Case $f(a) < 0$: The Intermediate Value Theorem guarentees a root in $(0, a)$.
Case $f(a) > 0$: Let $b = a - \frac{f(a)}{f'(a)}$. Note that $b - a = -\frac{f(a)}{f'(a)} > 0$.
According to Taylor's Theorem ($n = 1$), there exists a c in (a, b) such that

$$\begin{aligned} f(b) &= f(a) + f'(a)(b-a) + \frac{f''(c)}{2}(b-a)^2 \\ &= f(a) + f'(a)(-\frac{f(a)}{f'(a)}) + \frac{f''(c)}{2}(-\frac{f(a)}{f'(a)})^2 \\ &= \frac{f''(c)}{2}(\frac{f(a)}{f'(a)})^2 \end{aligned}$$

Since $f''(c) < 0$, $f(b) < 0$. The Intermediate Value Theorem guarentees a root in (a, b).

The number of positive roots is at least one and at most one. Therefore, f has exactly one positive root.

Question 5:

First, solve the system of equations to find the points of intersections.

$$\begin{aligned} y^2 &= 2x && (1) \\ x^2 + y^2 &= 8 && (2) \\ x^2 + 2x &= 8 \\ (x+1)^2 &= 9 \\ x + 1 &= 3 && \text{from (1), } x \geq 0 \\ x &= 2 \\ y &= 2 \text{ or } y = -2 \end{aligned}$$

The points of intersection are $(2, 2)$ and $(2, -2)$. Therefore, R is the region bounded by $x_1 = \frac{y^2}{2}$ and $x_2 = \sqrt{8 - y^2}$ on the interval $-2 \leq y \leq 2$.

(i) The area of R is given by

$$\int_{-2}^{2} x_2 dy - \int_{-2}^{2} x_1 dy = \int_{-2}^{2} \sqrt{8-y^2}dy - \int_{-2}^{2} \frac{y^2}{2} dy$$

For the first integral, let $y = 2\sqrt{2}\sin\theta$.

$$\begin{aligned}
\frac{dy}{d\theta} &= 2\sqrt{2}\cos\theta \\
\int_{-2}^{2} \sqrt{8-y^2}dy &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{8-8\sin^2\theta}(2\sqrt{2}\cos\theta)d\theta \\
&= 16\int_{0}^{\frac{\pi}{4}} \sqrt{1-\sin^2\theta}(\cos\theta)d\theta \\
&= 8\int_{0}^{\frac{\pi}{4}} 2\cos^2\theta d\theta \\
&= 8\int_{0}^{\frac{\pi}{4}} (\cos 2\theta + 1)d\theta \\
&= 8\left[\frac{\sin 2\theta}{2} + \theta\right]_{0}^{\frac{\pi}{4}} \\
&= 8\left(\frac{1}{2} + \frac{\pi}{4}\right) \\
&= 4 + 2\pi \\
\int_{-2}^{2} \sqrt{8-y^2}dy - \int_{-2}^{2} \frac{y^2}{2}dy &= (4+2\pi) - \left[\frac{y^3}{6}\right]_{-2}^{2} \\
&= (4+2\pi) - \frac{8}{3} \\
&= \frac{4}{3} + 2\pi
\end{aligned}$$

(ii) The volume of revolution is given by the integral

$$\begin{aligned}
\int_{-2}^{2} \pi(x_2^2 - x_1^2)dy &= \pi\int_{-2}^{2}((8-y^2) - \frac{y^4}{4})dy \\
&= \pi\left[8y - \frac{y^3}{3} - \frac{y^5}{20}\right]_{-2}^{2} \\
&= \frac{352}{15}\pi
\end{aligned}$$

Question 6:

(i) Integrate by parts. Using the substitution $u = 1 + x^2$, $\frac{du}{dx} = 2x$, consider the integral

$$\begin{aligned}\int \frac{x}{(1+x^2)^2} dx &= \int \frac{1}{2u^2} du \\ &= \frac{-1}{2u} + C \\ &= \frac{-1}{2(x^2+1)} + C \\ &= \frac{x^2}{2(x^2+1)} - \frac{1}{2} + C\end{aligned}$$

Using the results above,

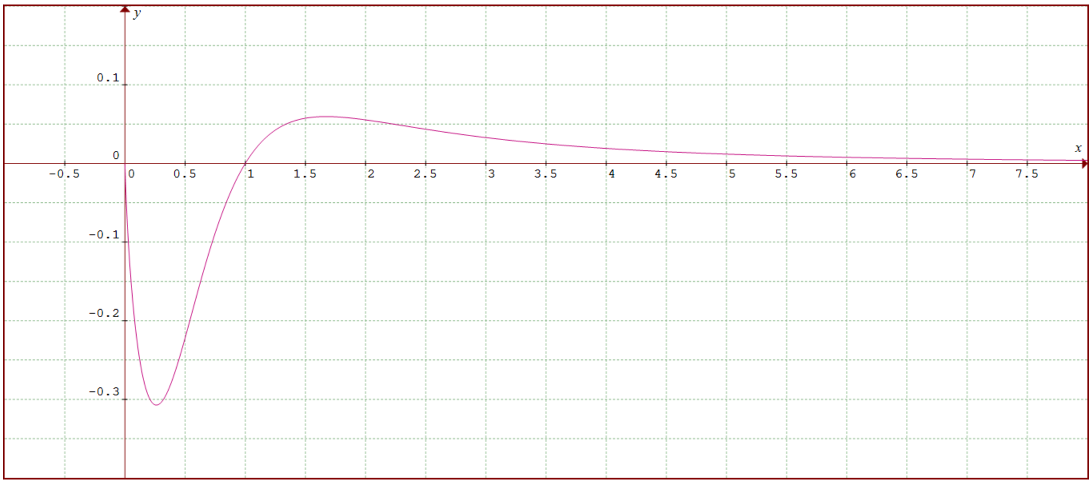
$$\begin{aligned}\int \frac{x \ln x}{(1+x^2)^2} dx &= \int (\ln x) \frac{d}{dx}\left(\frac{x^2}{2(x^2+1)}\right) dx \\ &= \frac{1}{2}\left[(\ln x)\frac{x^2}{x^2+1} - \int \frac{x^2}{x^2+1}\frac{d}{dx}(\ln x) dx\right] \\ &= \frac{x^2 \ln x}{2(x^2+1)} - \frac{1}{2}\int \frac{x^2}{x^2+1}\frac{1}{x} dx \\ &= \frac{x^2 \ln x}{2(x^2+1)} - \frac{1}{4}\int \frac{2x}{x^2+1} dx \\ &= \frac{x^2 \ln x}{2(x^2+1)} - \frac{1}{4}\ln(x^2+1) + C\end{aligned}$$

(ii) Let the antiderivative

$$\begin{aligned}F(x) &= \frac{x^2 \ln x}{2(x^2+1)} - \frac{1}{4}\ln(x^2+1) \\ &= \frac{1}{4}\left(\frac{\ln(x^2)}{1+x^{-2}} - \ln(x^2+1)\right) \\ &= \frac{1}{4}\left(\frac{\ln(\frac{x^2}{x^2+1})}{1+x^{-2}} - \frac{\ln(x^2+1)}{x^2+1}\right)\end{aligned}$$

$$\begin{aligned}\int_0^\infty \frac{x \ln x}{(1+x^2)^2} dx &= \lim_{b\to\infty} F(b) - \lim_{a\to 0^+} F(a) \\ &= \lim_{b\to\infty} \frac{1}{4}\left(\frac{\ln(\frac{b^2}{b^2+1})}{1+b^{-2}} - \frac{\ln(b^2+1)}{b^2+1}\right) - \lim_{a\to 0^+}\left(\frac{a^2 \ln a}{2(a^2+1)} - \frac{1}{4}\ln(a^2+1)\right) \\ &= \frac{1}{4}\lim_{b\to\infty} \frac{\ln(\frac{b^2}{b^2+1})}{1+b^{-2}} - \frac{1}{4}\lim_{b\to\infty}\frac{\ln(b^2+1)}{b^2+1} \\ &\quad - \lim_{a\to 0^+}\left(\frac{1}{2(a^2+1)}\right)\lim_{a\to 0^+}(a^2 \ln a) + \lim_{a\to 0^+}\frac{1}{4}\ln(a^2+1)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\frac{\ln 1}{1+0} - \frac{1}{4}\lim_{b\to\infty}\frac{\ln(b^2+1)}{b^2+1} - \left(\frac{1}{2}\right)\lim_{a\to 0^+}\left(\frac{\ln a}{a^{-2}}\right) + \frac{1}{4}\ln(0^2+1) \\
&= 0 - \frac{1}{4}\lim_{b\to\infty}\frac{\ln(b^2+1)}{b^2+1} - \frac{1}{2}\lim_{a\to 0^+}\frac{\frac{d}{da}(\ln a)}{\frac{d}{da}(a^{-2})} + 0 && \tfrac{0}{0} \text{ indeterminate form} \\
&= -\frac{1}{4}\lim_{b\to\infty}\frac{\frac{d}{db}(\ln(b^2+1))}{\frac{d}{db}(b^2+1)} - \frac{1}{2}\lim_{a\to 0^+}\frac{a^{-1}}{(-2)a^{-3}} && \tfrac{\infty}{\infty} \text{ indeterminate form} \\
&= -\frac{1}{4}\lim_{b\to\infty}\frac{\left(\frac{2b}{b^2+1}\right)}{2b} - \frac{1}{2}\lim_{a\to 0^+}\frac{-a^2}{2} \\
&= -\frac{1}{4}\lim_{b\to\infty}\frac{1}{b^2+1} - 0 \\
&= 0
\end{aligned}
$$



Graph of $y = \frac{x\ln x}{(1+x^2)^2}$

Alternative Solution:

$$\int_0^\infty \frac{x\ln x}{(1+x^2)^2}dx = \int_0^1 \frac{x\ln x}{(1+x^2)^2}dx + \int_1^\infty \frac{x\ln x}{(1+x^2)^2}dx$$

Extend the domain of the integrand to include zero and apply the substitution $u = \frac{1}{x}$ on the last integral.

Question 7:

(a) Differentiate y implicitly w.r.t. x.

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{x^2} - \frac{1}{z^2}\frac{dz}{dx}\\ -\frac{1}{x^2} - \frac{1}{z^2}\frac{dz}{dx} &= (\frac{1}{x}+\frac{1}{z})^2 - \frac{\frac{1}{x}+\frac{1}{z}}{x} - \frac{1}{x^2}\\ -\frac{1}{z^2}\frac{dz}{dx} &= (\frac{1}{x}+\frac{1}{z})^2 - \frac{1}{x}(\frac{1}{x}+\frac{1}{z})\\ &= \frac{1}{z^2}+\frac{1}{xz}\\ 0 &= \frac{dz}{dx}+1+\frac{z}{x}\end{aligned}$$

Using the substitution $u = zx$, $\frac{du}{dx} = z + \frac{1}{x}\frac{dz}{dx}$.

$$\begin{aligned}\frac{1}{x}\frac{du}{dx}+1=0\\ \frac{du}{dx} = -x\end{aligned}$$

When $x = 1$, $y = 2$, $z = 1$, $u = 1$.

$$\begin{aligned}u - 1 &= \int_1^x (-t)dt\\ u &= \frac{3-x^2}{2}\\ z &= \frac{3-x^2}{2x}\\ y &= \frac{1}{x}+\frac{2x}{3-x^2}\\ &= \frac{3+x^2}{x(3-x^2)}\end{aligned}$$

(b) Since height is always decreasing w.r.t. time, t is a function of h.

$$\frac{dt}{dh} = -\frac{4h-h^2}{\sqrt{h}}$$

The time taken to empty the sphere, $\tau = t(0) - t(4)$, can be derived using the Fundamental Theorem of Calculus.

$$\begin{aligned}\tau &= \int_4^0 \frac{dt}{dh}dh\\ &= \int_4^0 -\frac{4h-h^2}{\sqrt{h}}dh\end{aligned}$$

Using the substitution $u = \sqrt{h}$, $\frac{du}{dh} = \frac{1}{2\sqrt{h}}$,

$$\begin{aligned} -\int_4^0 \frac{4h-h^2}{\sqrt{h}}dh &= \int_0^4 \frac{4h-h^2}{\sqrt{h}}dh \\ &= \int_0^2 2(4u^2-u^4)du \\ &= 2\left[\frac{4u^3}{3}-\frac{u^5}{5}\right]_0^2 \\ &= \frac{128}{15} \end{aligned}$$

Question 8:

$|f'(x)| \leq M \implies -M \leq f'(x) \leq M$.

For $0 \leq x \leq 0.5$,

$$\begin{gathered} \int_0^x -Mdt \leq \int_0^x f'(t)dt \leq \int_0^x Mdt \\ -Mx \leq f(x) - f(0) \leq Mx \end{gathered}$$

Since $f(0) = 0$, $-Mx \leq f(x) \leq Mx$ and $|f(x)| \leq Mx$. Therefore,

$$\int_0^{0.5} |f(x)|dx \leq \int_0^{0.5} Mxdx = \frac{M}{8}$$

For $0.5 \leq x \leq 1$,

$$\begin{gathered} \int_x^1 -Mdt \leq \int_x^1 f'(t)dt \leq \int_x^1 Mdt \\ -M(1-x) \leq f(1) - f(x) \leq M(1-x) \end{gathered}$$

Since $f(1) = 0$, $-M(1-x) \leq f(x) \leq M(1-x)$ and $|f(x)| \leq M(1-x)$. Therefore,

$$\int_{0.5}^1 |f(x)|dx \leq \int_{0.5}^1 M(1-x)dx = \frac{M}{8}$$

Thus,

$$\int_0^1 |f(x)|dx = \int_0^{0.5} |f(x)|dx + \int_{0.5}^1 |f(x)|dx \leq \frac{M}{8} + \frac{M}{8} = \frac{M}{4}.$$