NUS
National University of Singapore

MA1102R – Calculus

Semester 2: AY2017/2018

Final Exam

9 May 2018 — Time allowed: 2 hours

<u>INSTRUCTIONS TO CANDIDATES</u>

1. Write your matriculation number clearly in the spaces provided on this cover page.
2. Please answer all the questions and write your solutions in the spaces inside this booklet.
3. You may use non-programmable calculators.
4. This paper contains 8 questions and comprises 15 pages.
5. Page 5 and Page 13 are left blank to provide sufficient space for solutions.
6. This is a closed book test. However, every candidate is allowed to bring one A4-size help sheet.

Matriculation Number:

A								

MARKS (Examiner's Use):

Q1	Q2	Q3	Q4	
Q5	**Q6**	**Q7**	**Q8**	**Total (Max. 70)**

Q1 Consider the function

$$f(x) = \sin^{-1} \frac{1}{\ln \dfrac{e^x + 1}{e^x - 1}}.$$

(i) Find the maximal domain of the function $f(x)$. [5 marks]

Q1 (continued)

(ii) Determine if $f(x)$ is a one-to-one function on its maximal domain. If yes, find the expression and the domain of definition for its inverse function $f^{-1}(x)$; if not, find x_1 and x_2 $(x_1 \neq x_2)$ such that $f(x_1) = f(x_2)$. [5 marks]

Q2 Let $f(x)$ be the function

$$f(x) = \begin{cases} \dfrac{a}{x-1}[3\sin(x-1) - 2\tan(\ln x)], & \text{if } 1/2 < x < 1, \\ b, & \text{if } x = 1, \\ \displaystyle\int_{4(x-1)}^{x^2} \mathrm{e}^{x+[\ln(t+1)]^c}\,\mathrm{d}t, & \text{if } x > 1. \end{cases}$$

Given that f is differentiable, find the values of a, b and c. [10 marks]

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Q3 Suppose the function $f(x)$ is continuous on $[0,1]$ and twice differentiable on $(0,1)$. Show that there exist $A \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $c \in (0,1)$ such that

$$f(1) = f(0) + f'\left(\frac{1}{2}\right) + Af''(c),$$

where f'' is the second-order derivative of f. [5 marks]

Q4 For given positive constants A and B, let

$$f(x) = (Ax^2 + B)^{-3/2}.$$

(i) Find

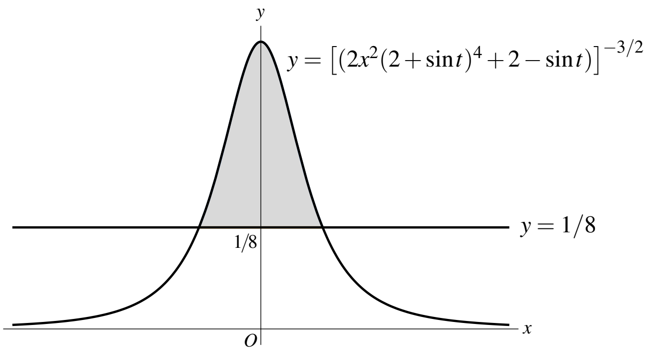
$$\int f(x)\,\mathrm{d}x.$$

[5 marks]

Q4 (continued)

(ii) For a given $t \in \mathbb{R}$, let $S(t)$ be the area bounded by the straight line $y = 1/8$ and the curve $y = \left[(2x^2(2+\sin t)^4 + 2 - \sin t)\right]^{-3/2}$ (the shaded region in the diagram). Show that

$$S(t) = \frac{1}{4(2-\sin t)\sqrt{2(2+\sin t)}}.$$ [5 marks]



Q4 (continued)

(iii) Find the absolute maximum and absolute minimum values of $S(t)$ when they exist. [5 marks]

Q5 For any non-negative integer n, let $I_n = \int_0^{\pi/2} \sin^n x \, dx$.

(i) Show that

$$I_{n+2} = \frac{n+1}{n+2} I_n$$

holds for any non-negative integer n. [5 marks]

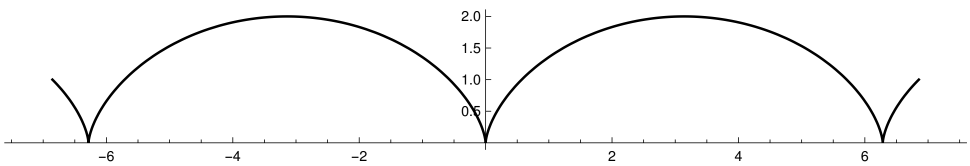
Q5 (continued)

(ii) Find the values of I_9 and I_{10}. [5 marks]

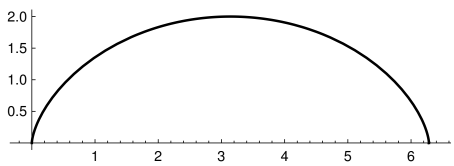
Q6 Consider the *cycloid* given by the parametric equations

$$\begin{cases} x = t - \sin t, \\ y = 1 - \cos t, \end{cases} \qquad t \in \mathbb{R},$$

whose graph is as follows:



Find the length of one period of this curve, which is a part of the above curve shown in the following the diagram:



[10 marks]

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Q7 Show that the equation

$$x = 2\tanh x$$

has exactly 3 solutions. [5 marks]

Q8 Solve the differential equation

$$(x^2y - y)\frac{\mathrm{d}y}{\mathrm{d}x} + (xy^2 + x) = 0, \quad -1 < x < 1; \quad y = 1 \text{ when } x = 0.$$

Express your answer in the form of $y = f(x)$. [5 marks]