NATIONAL UNIVERSITY OF SINGAPORE

MA1102R — CALCULUS

2018 – 2019 SEMESTER 1

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. Do not write your name.
2. This examination paper contains a total of **EIGHT (8)** questions and comprises **FOUR (4)** printed pages.
3. Answer **ALL** questions. The marks for questions are not necessarily the same; marks for each question are indicated at the beginning of the question.
4. Use a separate page for each question.
5. This is a **CLOSED BOOK** examination.
6. Two pieces of A4-size formula sheet are prepared and provided by the examiners.
7. Candidates may use non-graphing and non-programmable calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [8 marks]

For the function $f(x) = x(x-6)^{1/5}$,

(i) Find the open intervals on which it is increasing and decreasing;

(ii) Find the coordinates of all its local maximum and minimum points;

(iii) Find the open intervals on which it is concave up and concave down;

(iv) Find the coordinates of all its inflection points.

Question 2 [17 marks]

(a) Prove the following infinite limit using only the precise definition:

$$\lim_{x\to 1^+} \frac{x+3}{x-1} = \infty.$$

(b) Find the limit of

$$\lim_{x\to\infty} \left[\frac{1}{3}\left(3^{1/x} + 8^{1/x} + 9^{1/x}\right)\right]^x.$$

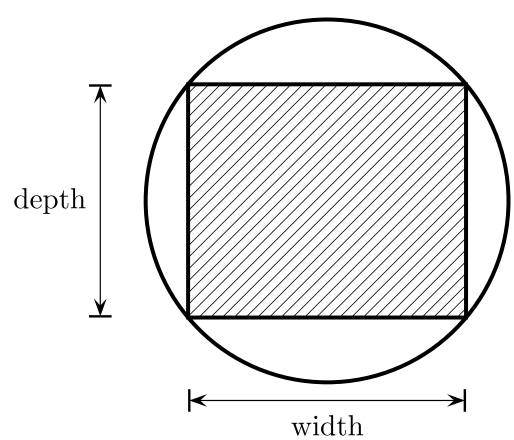
(c) Find the equation of the tangent line to the curve defined by

$$y = (1/x)^{\ln x}$$

at $x = e$.

Question 3 [8 marks]

The stiffness of a wooden beam of rectangular cross section is proportional to the product of the width and the cube of the depth of the cross section. Find the width and depth of the stiffest beam that can be cut out of a circular log of radius R. Justify your answers.



Question 4 [8 marks]

Let f be a one-to-one function on an open interval I and g be the inverse function of f.

(i) If f is twice differentiable and $f'(x) \neq 0$ for every $x \in I$, show that for any y in the domain of g,

$$g''(y) = -\frac{f''(g(y))}{[f'(g(y))]^3}.$$

(ii) If f is three times differentiable and $f'(x) \neq 0$ for every $x \in I$, show that for any y in the domain of g,

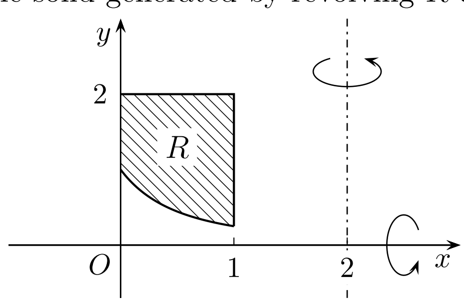
$$g'''(y) = \frac{3[f''(g(y))]^2 - f'(g(y))f'''(g(y))}{[f'(g(y))]^5}.$$

Question 5 [19 marks]

(a) Let R be the region enclosed by the curve $y = \dfrac{1}{(1+x)^2}$ and the lines $y = 2$, $x = 0$ and $x = 1$.

(i) Find the volume of the solid generated by revolving R about the x-axis.

(ii) Find the volume of the solid generated by revolving R about the line $x = 2$.



(b) Let f be a nonnegative differentiable function such that f' is continuous on $[0, \infty)$. Suppose that the length of the curve $y = f(x)$ from $x = 0$ to $x = b$ is $b + \frac{2}{3}b^3$ for every $b > 0$, and that $f(0) = \frac{2}{3}$. Find the expression of $f(x)$ for $x \geq 0$.

Question 6 [15 marks]

(a) Find the indefinite integral

$$\int \frac{x}{\sqrt{9+8x^2-x^4}}\,dx.$$

(b) Find all the values of p for which the integral

$$\int_0^1 x^p \ln x\,dx$$

converges, and evaluate the integral for these values of p.

Question 7 [15 marks]

(a) Solve the initial value problem

$$\frac{dy}{dx}+\frac{2x+1}{x}\,y=2x \qquad (x>0),$$

such that $y=1$ when $x=1$.

(b) The rate at which a drug is absorbed into the bloodstream is modeled by the first-order differential equation

$$\frac{dQ}{dt}=a-bQ,$$

where Q denotes the concentration of drug in the bloodstream at time t, and a and b are positive constants. Assume that no drug is initially present in the bloodstream.

(i) Find the limiting concentration of the drug in the bloodstream as $t\to\infty$.

(ii) How long does it take for the concentration to reach half of the limiting value?

Question 8 [10 marks]

(a) Let f be a function which is continuous on $[0,1]$ and twice differentiable on $(0,1)$. If $f(0)=f(1)=0$ and $f(a)>0$ for some $a\in(0,1)$, prove that there exists $c\in(0,1)$ such that $f''(c)<0$.

(b) Let g be a differentiable function such that g' is continuous on $[0,1]$, and let M be the maximum value of $|g'(x)|$ on $[0,1]$, prove that

$$g(1/2)\le\frac{M}{4}+\int_0^1 g(x)\,dx.$$