Question 1

Part A

Defining T as tension in string, $m_1 = 4.00\,\mathrm{kg}$ and $m_2 = 6.00\,\mathrm{kg}$ as masses of objects 1 and 2, a_1 and a_2 as accelerations of objects 1 and 2.

$$T = m_1 a_1 \tag{1a}$$
$$m_2 g - T = m_2 a_2 \tag{1b}$$
$$a_1 = a_2 \tag{1c}$$

From above,

$$m_2 g = m_2 a + m_1 a$$

$$\boxed{a = \frac{m_2 g}{m_1 + m_2} \approx 5.88\,\mathrm{m\,s^{-2}}}$$

$$\boxed{T = \frac{m_1 m_2}{m_1 + m_2} g \approx 23.5\,\mathrm{N}}$$

Part B

$$m_2 g - 2T = m_2 a_2 \tag{2a}$$
$$m_1 a_1 = T \tag{2b}$$
$$a_1 = 2a_2 \tag{2c}$$

From above,

$$m_2 g = m_2 a_2 + 4m_1 a_2$$

$$\boxed{a_2 = \frac{1}{2}\frac{m_2 g}{m_2 + 4m_1} \approx 2.67\,\mathrm{m\,s^{-2}}}$$

$$\boxed{a_1 \approx 5.35\,\mathrm{m\,s^{-2}}}$$

$$\boxed{T \approx 21.4\,\mathrm{N}}$$

Question 2

Part A

Defining U as upthrust, M as total mass of balloon and helium, $\rho_a = 1.20\,\mathrm{kg\,m^{-3}}$ as density of air, $\rho_{\mathrm{He}} = 0.178\,\mathrm{kg\,m^{-3}}$ as density of helium, $V = 0.0045\,\mathrm{m^3}$ as volume of balloon, $m = 0.0025\,\mathrm{kg}$ as mass of deflated balloon.

$$\begin{aligned} F_{\mathrm{net}} &= U - Mg \\ &= \rho_a V g - (m + \rho_{\mathrm{He}} V) g \\ \therefore \quad a &= \frac{\rho_a V - m - \rho_{\mathrm{He}} V}{m + \rho_{\mathrm{He}} V} g \end{aligned}$$

From kinematics,

$$s = \frac{1}{2}at^2$$
$$t = \sqrt{\frac{2s}{a}}$$

Putting everything together gives $\boxed{t \approx 0.694\,\mathrm{s}}$.

Part B

Speed of balloon immediately before string is taut:

$$v = at \approx 4.323\,\mathrm{m\,s^{-1}}$$

By Conservation of Momentum and defining $m_s = 0.015$ kg as mass of stone,

$$Mv = (M + m_s)v_f$$

$$\boxed{v_f = \frac{M}{M + m_s}v \approx 0.780\,\mathrm{m\,s^{-1}}}$$

Question 3

Defining G as the gravitational constant, $M = 5.98\times10^{24}$ kg as the mass of the Earth, m as the mass of the meteoroid, $r_1 = 2.8\times10^8$ m as the distance between the meteoroid and the Earth initially and $r_2 = 8.5\times10^6$ m as the distance between the meteoroid and the Earth at the closest approach, $v_1 = 1.1 \times 10^3\,\mathrm{m\,s^{-1}}$ as the initial speed of the meteoroid and v_2 as the speed of the meteoroid at closest approach.

Part A

By Conservation of Energy,

$$\frac{1}{2}mv_1^2 - \frac{GMm}{r_1} = \frac{1}{2}mv_2^2 - \frac{GMm}{r_2}$$
$$v_2 = \sqrt{2\left[\frac{v_1^2}{2} - GM\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\right]}$$

$$\boxed{v_2 \approx 9.60 \times 10^3\,\mathrm{m\,s^{-1}}}$$

Part B

As the total energy of the system, i.e. $\frac{1}{2}mv^2 - \frac{GMm}{r}$, is less than zero, the system is bound — the meteoroid does not have sufficient energy to escape Earth's gravitational field. Hence it will return to Earth's vicinity.

Question 4

Defining k as the wave number in $\mathrm{rad\,m^{-1}}$, ω as the angular frequency of the wave, v as the propagation speed of the wave, T as the tension in the cord, $\mu_1 = 0.10\,\mathrm{kg\,m^{-1}}$ and $\mu_2 = 0.20\,\mathrm{kg\,m^{-1}}$ as the linear densities of the cord at different sections, f as the frequency of the wave, λ_1 and λ_2 as the wavelengths of the wave at the first and second section of the cord respectively.

Part A

$$\boxed{k = \frac{2\pi}{\lambda} \implies \lambda = \frac{2\pi}{k} \approx 1.0\,\mathrm{m}}$$

Part B

$$v = \frac{\omega}{k} \tag{3a}$$
$$v = \sqrt{\frac{T}{\mu}} \tag{3b}$$

From the above,

$$\boxed{T = \mu\left(\frac{\omega}{k}\right)^2 \approx 0.90\,\mathrm{N}}$$

Part C

$$v = \lambda f \tag{4a}$$
$$T = v^2\mu \tag{4b}$$

From the above, and from the fact that the tension is the same throughout the string,

$$\mu_1(\lambda_1 f)^2 = \mu_2(\lambda_2 f)^2$$
$$\mu_1\lambda_1^2 = \mu_2\lambda_2^2$$
$$\lambda_2 = \sqrt{\frac{\mu_1}{\mu_2}}\cdot\lambda_1$$

$$\boxed{\lambda_2 \approx 0.74\,\mathrm{m}}$$

Note that the frequency of the wave does not change in order for the wave to be continuous.

Note 1. *Don't forget to use the more precise value of $\lambda_1 \approx 1.047$ for this part, otherwise your answer (0.71 m) will be off by 4%!*

Question 5

Defining L_{crouch} and L_{stand} as the angular momentum of the girl when she is crouching and standing respectively, m as the mass of the girl, v_{crouch} and v_{stand} as the velocity of the girl at points B and C respectively, r_{crouch} and r_{stand} as the distance between the centre of the mass of the girl from the pivot at when she is crouching and standing respectively, Δh as the change in height of her centre of mass.

By Conservation of Angular Momentum,

$$L_{\mathrm{crouch}} = L_{\mathrm{stand}}$$
$$mv_{\mathrm{crouch}}r_{\mathrm{crouch}} = mv_{\mathrm{stand}}r_{\mathrm{stand}}$$
$$v_{\mathrm{crouch}}\cdot 3.7 = v_{\mathrm{stand}}\cdot(3.7-0.6)$$
$$v_{\mathrm{crouch}} = \frac{3.1}{3.7}v_{\mathrm{stand}}$$

By Conservation of Energy,

$$\frac{1}{2}mv^2 = mg\Delta h$$
$$v^2 = 2g\Delta h$$

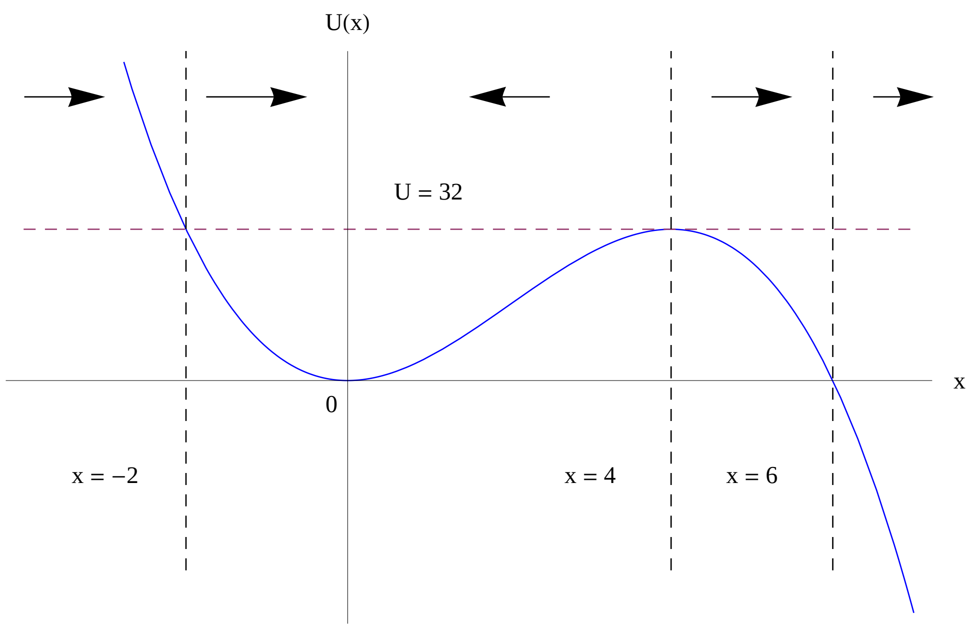
Hence with the above results

$$2g\Delta h_{\text{crouch}} = 2g\Delta h_{\text{stand}}\left(\frac{31}{37}\right)^2$$
$$\Delta h_{\text{stand}} = \Delta h_{\text{crouch}}\left(\frac{37}{31}\right)^2$$
$$h - 1.2 = (1.2 - 0.6)\left(\frac{37}{31}\right)^2$$

$$\boxed{h \approx 2.05\,\mathrm{m}}$$

Question 6

Part A



Arrows indicate direction of force within each respective regions.

Find critical points by:

$$\frac{d}{dx}U(x) = 0$$
$$12x - 3x^2 = 0$$
$$x(4 - x) = 0$$

$$\therefore \quad x = 0 \quad \text{or} \quad x = 4$$

Find roots by:

$$
\begin{aligned}
U(x) &= 0 \\
6x^2 - x^3 &= 0 \\
x^2(6 - x) &= 0
\end{aligned}
$$

$$\therefore \quad x = 0 \quad \text{or} \quad x = 6$$

Particle with energies below 32 within the region $-2 < x < 4$ will oscillate within the regions. Particles with energies greater than 32 or particles outside the region $-2 < x < 4$ will eventually move in the positive x direction forever.

$x = 0$ is a stable equilibrium point; $x = 4$ is an unstable equilibrium point.

Part B

Section 1

Defining v_{bottom} to be speed of object at the bottom and by Conservation of Energy,

$$
\begin{aligned}
\frac{1}{2}mv_0^2 + mgh &= \frac{1}{2}mv_{\text{bottom}}^2 \\
v_0^2 + 2gh &= v_{\text{bottom}}^2 \\
v_{\text{bottom}} &\approx 6.943\,\text{m}\,\text{s}^{-1}
\end{aligned}
$$

Time taken for object to stop when moving from bottom to maximum displacement along Plane B,

$$
\begin{aligned}
v_{\text{bottom}} &= at_B \\
t_B &= \frac{v_{\text{bottom}}}{a} \\
&= \frac{v_{\text{bottom}}}{g\sin(15^\circ)} \\
&\approx 2.737\,\text{s}
\end{aligned}
$$

Time taken for object to stop when moving from bottom to maximum displacement along Plane A,

$$
\begin{aligned}
v_{\text{bottom}} &= at_A \\
t_A &= \frac{v_{\text{bottom}}}{a} \\
&= \frac{v_{\text{bottom}}}{g\sin(25^\circ)} \\
&\approx 1.676\,\text{s}
\end{aligned}
$$

Period is then

$$\boxed{T = 2(t_A + t_B) \approx 8.83\,\text{s}}$$

Section 2

Maximum Height Using Conservation of Energy, we consider the energy of the particle when it reaches the bottom, where we define $v = 3.00\,\text{m}\,\text{s}^{-1}$ and $h = 2.00\,\text{m}$:

$$
\begin{aligned}
E &= \frac{1}{2}mv^2 + mgh + fs \\
&= \frac{1}{2}mv^2 + mgh + \mu_k mg\cos(25^\circ)\,\frac{h}{\sin(25^\circ)}
\end{aligned}
$$

We equate the above energy to the energy of the particle when it is at its maximum height on Plane B:

$$\begin{aligned}\frac{1}{2}mv^2 + mgh + \mu_k mg\cos(25^\circ)\,\frac{h}{\sin(25^\circ)} &= mgh' + f's' \\ &= mgh' + \mu_k mg\cos(15^\circ)\,\frac{h'}{\sin(15^\circ)}\end{aligned}$$

$$\boxed{h' = 1.99\,\mathrm{m}}$$

Permanent Stop Considering whether static friction will stop subsequent motion on Plane B,

$$mg\sin(15^\circ) > \mu_s mg\cos(15^\circ)$$

Hence static friction will not stop subsequent motion on Plane B.

Considering whether static friction will stop subsequent motion on Plane A,

$$mg\sin(25^\circ) > \mu_s mg\cos(25^\circ)$$

Hence static friction will not stop subsequent motion on Plane A.

This suggests that object will continue to move back down upon reaching its maximum, albeit decaying, height on either planes. It will only come to a permanent stop at O — the bottom — when it has exhausted all its energy to friction.

Question 7

Part A

Section 1

Defining z to be position of center of mass from the bottom,

$$\begin{aligned}(3m+m)z &= 3m\cdot 3b + m\cdot b \\ z &= \frac{5}{2}b\end{aligned}$$

$$\boxed{\therefore \quad \Delta z = 3b - z = \frac{b}{2}}$$

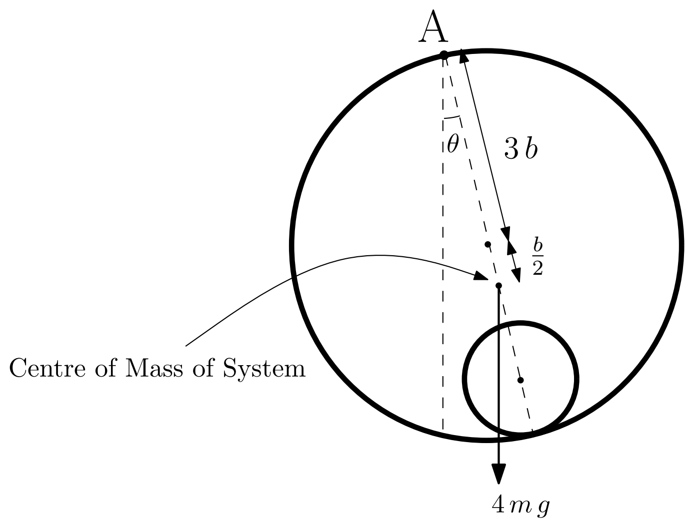
Section 2

Defining I_{net} to be the moment of inertia about the centre of mass of the system (the two hoops combined),

$$\begin{aligned}I_{\text{net}} &= I_{\text{3m}} + I_{\text{m}} \\ &= \left[3m\cdot(3b)^2 + 3m\cdot\left(\frac{b}{2}\right)^2\right] + \left[m\cdot b^2 + m\cdot\left(\frac{5b}{2} - b\right)^2\right]\end{aligned}$$

$$\boxed{I_{\text{net}} = 31mb^2}$$

Part B



Analysing torque about Point A and using the parallel axis theorem,

$$\tau = I\ddot{\theta}$$
$$-(4mg)\cdot\left(3b+\frac{b}{2}\right)\cdot\sin(\theta) = \left[I_{\text{net}} + 4m\left(3b+\frac{b}{2}\right)^2\right]\ddot{\theta}$$
$$-14mgb\sin(\theta) = 80mb^2\ddot{\theta}$$

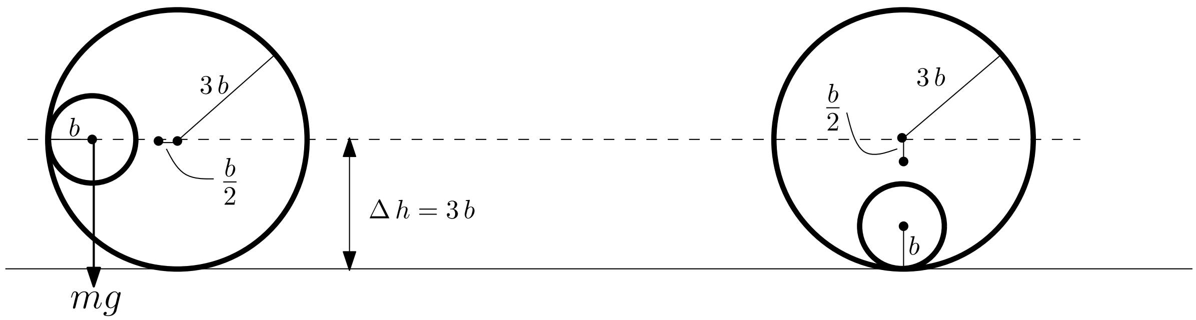
Using small angle approximation ($\sin(\theta) \approx \theta$),

$$\ddot{\theta} = -\frac{7g}{40b}\theta$$
$$\implies \quad \omega^2 = \frac{7g}{40b}$$
$$\frac{2\pi}{T} = \sqrt{\frac{7g}{40b}}$$

$$\boxed{T = 4\pi\sqrt{\frac{10b}{7g}}}$$

Part C

When the centres of the hoops lie in a vertical line, the point where the two hoops meet is in contact with the table. We need to find the KE of the hoop system. We shall do this by looking at the instant where the hoop system seems to revolve around the contact point with the table. The KE will be given by the moment of inertia about that point, and the angular velocity about that point.



At point of release

When centre of both hoops form a vertical line

By Conservation of Energy, and finding the moment of inertia about the point where the hoops meet,

$$\begin{aligned} mg\Delta h &= \frac{1}{2} I\omega^2 \\ &= \frac{1}{2}\left[I_{\text{net}} + 4m \cdot \left(\frac{5b}{2}\right)^2\right]\omega^2 \\ &= 28mb^2\omega^2 \\ mg(3b-b) &= 28mb^2\omega^2 \\ \omega &= \sqrt{\frac{g}{14b}} \end{aligned}$$

Using the fact that speed of the center of mass of big hoop is $v = r\omega = 3b\omega$,

$$\boxed{v = \sqrt{\frac{9gb}{14}}}$$

Note 2. *It is not appropriate to find the KE by taking the moment of inertia of the hoop system about its center of mass and the angular velocity about the point of contact with the table. This is because the center of mass of the hoop system is executing rotation about some other point. So unless you take that rotational KE of the moving center of mass into account, you should not perform the said approach.*

Part D

By Conservation of Energy, and using the fact that $v = r\omega = b\omega$,

$$\begin{aligned} mg\Delta h &= \frac{1}{2}mv^2 + \frac{1}{2}I_{\text{m}}\omega^2 \\ &= \frac{1}{2}mv^2 + \frac{1}{2}(mb^2)\omega^2 \\ &= mv^2 \\ mg(3b-b) &= mv^2 \end{aligned}$$

$$\boxed{v = \sqrt{2gb}}$$

Question 8

Part A

Section 1

Defining h to be the height of the river from the river bed,

$$\begin{aligned} \mathrm{d}F &= P \cdot \mathrm{d}A \\ &= \rho g(h - h')l\,\mathrm{d}h' \\ \int \mathrm{d}F &= \int_0^h \rho g(h - h')l\,\mathrm{d}h' \end{aligned}$$

$$\boxed{F = \frac{1}{2}\rho g l h^2}$$

Section 2

$$\begin{aligned} \mathrm{d}\tau &= h'\,\mathrm{d}F \\ &= h'\rho g(h - h')l\,\mathrm{d}h' \\ \int \mathrm{d}\tau &= \int_0^h h'\rho g(h - h')l\,\mathrm{d}h' \end{aligned}$$

$$\boxed{\tau = \frac{1}{6}\rho g l h^3}$$

Section 3

$$\begin{aligned} \tau &= RF \\ R &= \frac{\frac{1}{6}\rho g l h^3}{\frac{1}{2}\rho g l h^2} \end{aligned}$$

$$\boxed{R = \frac{h}{3}}$$

Part B

Section 1

By Conservation of Momentum, and the equality of the approach speed and separation speed ($v_0 = v_f - v'$), speed of αM after collision is:

$$\begin{aligned} Mv_0 &= Mv' + \alpha M v_f \\ &= M(v_f - v_0) + \alpha M v_f \\ 2v_0 &= (1 + \alpha)v_f \\ v_f &= \frac{2}{1 + \alpha}v_0 \end{aligned}$$

Speed of M after bouncing away from wall is:

$$\begin{aligned} v'' &= -v' \\ &= v_0 - \frac{2}{1+\alpha} v_0 \\ &= \frac{\alpha - 1}{\alpha + 1}\, v_0 \end{aligned}$$

We find that for if only one collision is possible, v_f must be greater than v'', this gives us

$$\begin{aligned} \frac{2}{1+\alpha} &\geq \frac{\alpha-1}{\alpha+1} \\ \alpha &\leq 3 \end{aligned}$$

Hence for $\alpha \leq 3$, we find that mass M cannot catch up with mass αM for a second and subsequent collisions, thus only one collision is possible.

Section 2

After the first collision, we have

$$v_f = \frac{2}{5} v_0 \quad \& \quad v'' = \frac{3}{5} v_0$$

Using a similar approach as per the previous section, we have the speed of αM after the second collision to be:

$$\begin{aligned} \frac{3}{5} M v_0 + \frac{2}{5} \alpha M v_0 &= M u' + \alpha M u_f \\ &= M(u_f - \frac{1}{5} v_0) + \alpha M u_f \\ u_f &= \frac{4 + 2\alpha}{5 + 5\alpha} v_0 = \frac{12}{25} v_0 \end{aligned}$$

Speed of M after colliding with αM the second time is:

$$\begin{aligned} u'' &= u_f - (v'' - v_f) \\ &= \frac{12}{25} v_0 - \frac{1}{5} v_0 \\ &= \frac{7}{25} v_0 < u_f \end{aligned}$$

Hence mass M is moving in the same direction as mass αM, but with a slower speed. There will be no more collisions. The final speeds of the blocks are then $\boxed{\frac{12}{25} v_0 \text{ for mass } \alpha M}$ and $\boxed{\frac{7}{25} v_0 \text{ for mass M}}$.