Suggested Solutions for PC1141 AY2007/2008 Semester 1

NUS Physics Society

Part I

1. Taking the motion to the right as positive and downwards motion as negative,

$$v\cos 30^\circ = \frac{s}{t} = \frac{1}{t} \quad (1)$$

$$-s = -v(\sin 30^\circ)t - \frac{1}{2}gt^2 \quad (2)$$

Hence, $1 = v(\sin 30^\circ)t + \frac{1}{2}gt^2$, and $v(\cos 30^\circ)t = 1$

Rearranging and dividing,

$$\begin{aligned}\frac{v(\sin 30^\circ)t}{v(\cos 30^\circ)t} &= 1 - \frac{1}{2}gt^2 \\ \text{Hence, } \tan 30^\circ = 0.5774 &= 1 - \frac{1}{2}(9.80)(t^2) \\ \text{Hence, } t &= 0.2936\text{s}\end{aligned}$$

Substituting $t = 0.2936$s into Eqn. (1), $v\cos 30^\circ = \frac{1}{0.2936} = 0.8660v$. Hence, v=3.933 ms^{-1}.

From the conservation of energy, $\frac{1}{2}mv^2 = mgh$. Hence, $h = 0.7892$m

2. (a) Please refer to the diagram on the next page.

   i. $f_1 = \mu_k mg$, $N_1 = mg$

   ii. $f_2 \leq \mu_s Mg$, $N_2 = Mg$

   (b) The equations are $F - f_1 = ma$ and $f_2 - F = Ma$

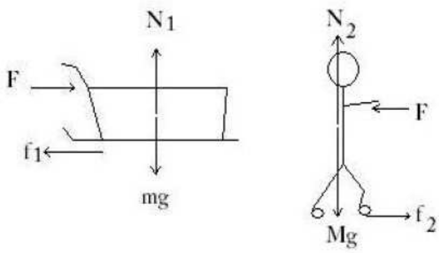

Figure 1: Forces on the sledge and the person for Question 2(a)(i) and 2(a)(ii)

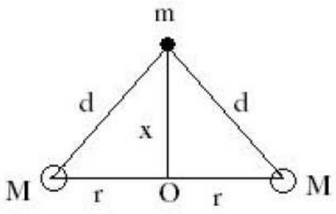

Figure 2: Reference diagram for Questions (6)(b) to (6)(e)

Since $T_1 = mg$, 300Hz (2L)$= \sqrt{\frac{mg}{\mu}}$. From $\rho = \frac{m}{V}$,

$$600\text{Hz } (L) = \sqrt{\frac{\rho_{Al} V_{Al} g}{\mu}} = \sqrt{\frac{2700(V_{Al} g)}{\mu}} \text{m Hz}$$

After half of the load is submerged in water, we have the tension on the string $T_2 = mg - B$. If $f' =$ the fundamental frequency after the submersion, then $2f'L = \sqrt{\frac{T_2}{\mu}}$. From $B = \rho_w V_w g$ and $V_w = V_{Al}/2$,

$$T_2 = \rho_{Al} V_{Al} g - \rho_w V_w g = [(2700 V_{Al}) - (1000 V_w)]g = 2200(V_{Al} g)$$

$$\text{Hence, } 2f'L = \sqrt{\frac{2200 V_{Al} g}{\mu}}, \text{ and } \frac{2f'L}{600\text{Hz}(L)} = \sqrt{\frac{2.20}{2.70}}$$

$$2f' = 0.9027 \times 600\text{Hz}, \text{ and } f' = 270.8\text{Hz} = 271\text{Hz}$$

Part II

6. (a) For the binary stars, gravitational force = centripetal force

$$\frac{GM^2}{(2r)^2} = Mr\omega^2 = \frac{GM^2}{4r^2} = r\left(\frac{2\pi}{T}\right)^2$$

$$\text{Hence, } T = 4\pi\sqrt{\frac{r^3}{GM}}$$

(b) Let d= distance of each of the binary stars to m, and gravitational potential $U = \frac{-GMm}{d}$. Given, $d = \sqrt{x^2 + r^2}$

$$\text{Gravitational potential energy of any one of the binary stars on m, } U_{Mm} = \frac{-GMm}{\sqrt{x^2 + r^2}}$$

$$\text{Hence, total gravitational potential energy, } U_{total} = 2\left(\frac{-GMm}{\sqrt{x^2 + r^2}}\right)$$

(c) Taking the upwards and the right directions as positive (as shown in diagram),

$$\text{the force of } M_A \text{ on m, } \vec{F_A} = \frac{-GMm}{d^2}(\cos\theta\vec{j} + \sin\theta\vec{i})$$
$$\text{the force of } M_B \text{ on m, } \vec{F_B} = \frac{-GMm}{d^2}(\cos\theta\vec{j} - \sin\theta\vec{i})$$

Hence, $\cos\theta = \frac{x}{d} = \frac{x}{\sqrt{x^2+r^2}}$.

$$\vec{F} = \vec{F_A} + \vec{F_B} = \frac{-2GMm}{d^2}\cos\theta\vec{j} = \frac{-2GMmx}{\sqrt{(x^2+r^2)^3}}\vec{j}$$

The negative sign indicates that the attraction is in the downwards direction, towards O. Hence, $|\vec{F}| = \frac{2GMmx}{\sqrt{(x^2+r^2)^3}}$

(d) i. For $x \gg r$, $x^2 + r^2 \approx r^2$

$$\text{Since } U = \frac{-2GMm}{\sqrt{(x^2+r^2)}} \text{ and } |\vec{F}| = \frac{2GMmx}{\sqrt{(x^2+r^2)^3}},\ U \approx \frac{-2GMm}{x} \text{ and } |\vec{F}| \approx \frac{2GMmx}{x^2}.$$

ii. When $x = 0$, $x^2 + r^2 = r^2$. Hence, $U = \frac{-2GMm}{r}$ and $|\vec{F}| = 0$.

(e) $F = \frac{2GMmx}{\sqrt{(x^2+r^2)^3}}$. Given $F = ma = mx\omega^2$, $a = \frac{2GMx}{\sqrt{(x^2+r^2)^3}} = -\omega^2 x$. Since $r \gg x$, $a \approx \frac{-2GMx}{(\sqrt{r^2})^3} = -\omega^2 x$ (S.H.M.)

$$\text{Hence, } \omega^2 = \frac{2GM}{r^3} = \left(\frac{2\pi}{T}\right)^2, \text{ and rearranging, } T = 2\pi\sqrt{\frac{r^3}{2GM}}$$

7. (a) Surface area, A = surface area of endcaps + surface area of curved portion $= A_1 + A_2$.

$$\begin{aligned} A_1 &= 2(\text{area of each endcap}) = 2\pi R^2, \text{ while } A_2 = 2\pi R \times R = 2\pi R^2 \\ \text{Hence, } \frac{A_1}{A_2} &= \frac{1}{2} \end{aligned}$$

Since the can has uniform thickness, the distribution of mass is also uniform. Hence, mass at endcaps, $m_1 = \frac{A_1}{A}(M) = \frac{1}{2}M$ and mass of curved portion, $m_2 = \frac{A_2}{A} = \frac{1}{2}M$.

Since the endcaps can be treated as disks, $I_1 = \frac{1}{2}m_1R^2$. Hence, $I_1 = \frac{1}{2}\left(\frac{1}{2}M\right)R^2 = \frac{1}{4}MR^2$.

For the curved portion, given that $R \gg t$, it can be treated as a cylindrical shell. Hence, $I_2 = m_2R^2$, and the inertia of the whole object, $I = I_1 + I_2 = \frac{3}{4}MR^2$.

(b) Referring to the diagram shown,

$$\begin{aligned} Mg\sin 20^\circ - f &= Ma \\ R \times f = I\alpha, \text{ with } a &= R\alpha. \text{ Hence, } R \times f = I\left(\frac{a}{R}\right) \\ \text{Given that } I = \frac{3}{4}MR^2,\ R \times f &= \frac{3}{4}MR^2\left(\frac{a}{R}\right) = \frac{3}{4}MRa. \text{ Hence, } f = \frac{3}{4}Ma \end{aligned}$$

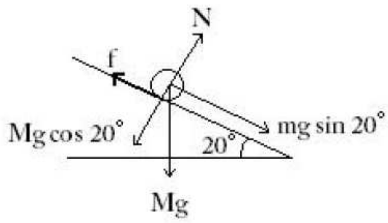

Figure 3: Diagram for Question 7(b)

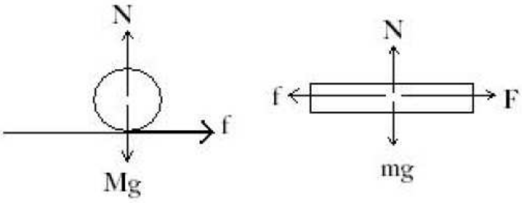

Figure 4: Diagram for Question 7(c)

Substituting $f = \frac{3}{4}Ma$ into $Mg\sin 20° - f = Ma$:

$$Mg\sin 20° - \frac{3}{4}Ma = Ma. \text{ Hence, } g\sin 20° = \frac{7}{4}Ma, \text{ and } a = 1.915\text{ms}^{-2}$$
$$\text{Using } s = ut + \frac{1}{2}at^2, \text{ where, } u = 0, s = \frac{1}{2}at^2 = 2m, \ 2m = \frac{1}{2}(1.915\text{ms}^-2)(t^2)$$
$$\text{Hence, } t = 1.445\text{s} = 1.45\text{s}$$

(c) Note: the can is rolling backwards with respect to the block.

$$\text{For M, } f = Ma_1, Rf = I\alpha, \text{ while for m, } F - f = ma_2. \text{ Together, } a_2 - a_1 = R\alpha$$
$$\text{Solving, } Rf = I\alpha = RMa_1, \text{ where } I = \frac{3}{4}MR^2. \text{ Hence, } RMa_1 = \frac{3}{4}MR^2\alpha \text{ and } a_1 = \frac{3}{4}R\alpha$$
$$\text{From } R\alpha = a_2 - a_1, a_1 = \frac{3}{4}(a_2 - a_1). \text{ Hence, } 7a_1 = 3a_2$$
$$\text{From } f = Ma_1, f = M\left(\frac{3}{7}a_2\right)$$
$$\text{Hence, } F - f = ma_2 = F - \frac{3}{7}Ma_2 = ma_2$$
$$\text{Hence, } F = \left(\frac{3}{7}M + m\right)a_2, \text{ and } a_2 = \frac{7F}{3M + 7m}$$
$$\text{From } a_1 = \frac{3}{7}a_2, a_1 = \frac{3F}{3M + 7m}$$
$$\text{Since } a_2 - a_1 = R\alpha, \alpha = (a_2 - a_1)\left(\frac{1}{R}\right) = \frac{4F}{(3M + 7m)R}$$

8. (a) i. Using the conservation of momentum,

$$m_A v_0 = m_C v_f(\cos 30^\circ) + m_B v_f(\cos 30^\circ) + m_A v_A,$$

where $v_A =$ final velocity of m_A. Also from the conservation of momentum,

$$0 = m_A v_A \sin\alpha + m_B v_f \sin 30^\circ - m_C v_f \sin 30^\circ$$

where α is the angle between v_A and the horizontal (as shown in the question paper).

$$\text{Hence, } \alpha = 0^\circ \text{ or } \alpha = 180^\circ$$

This means that object A will travel along the horizontal after the collision. Given, $m_A = m_B = m_C$:

$$v_0 = 2v_f \cos 30^\circ + v_A = 10\text{ms}^{-1} = 1.732v_f + v_A, \text{ and } v_A = 10\text{ms}^{-1} - 1.732v_f$$

From the conservation of energy, $\frac{1}{2}m_A(v_0)^2 = \frac{1}{2}m_B v_f^2 + \frac{1}{2}m_C v_f^2 + \frac{1}{2}m_A(v_A)^2$, which can be simplified to be $v_0^2 = 2v_f^2 + v_A^2 = (10\text{ms}^{-1})^2$.
Substituting $v_A = 10\text{ms}^{-1} - 1.732v_f$ into $v_A^2 = 100\text{m}^2\text{s}^{-2} - 2v_f^2$:

$$(10\text{ms}^{-1} - 1.732v_f)^2 = 100\text{m}^2\text{s}^{-2} - 2v_f^2 = 100\text{m}^2\text{s}^{-2} - 10\text{ms}^{-1}(2)(1.732v_f) + 3v_f^2$$
$$\text{Hence, } -34.64\text{ms}^{-1} = -5v_f, \text{ and } v_f = 6.928\text{ms}^{-1} = 6.93\text{ms}^{-1}$$

ii. From 8(a)(i), $v_A = 10\text{ms}^{-1} - 1.732v_f$, $v_A = -2.00\text{ ms}^{-1}$, since $v_f = 6.93\text{ms}^{-1}$. Hence, the direction of v_A is in the negative x-direction (as indicated by the negative sign), if the direction of v_0 is taken to be positive.

(b) i. For elastic collision, the relative velocities, v has the relationship $v_{approach} = v_{separation}$.

$$v_2 - v_1 = v_0 \quad (3)$$
$$v_3 - v_2' = v_2 \quad (4)$$

where, v_1 =final velocity of m_1 after collision, v_2 =final velocity of m_2 after first collision (and also the initial velocity for the second collision), v_2' =final velocity of m_2 after second collision, and v_3 = final velocity of m_3 after second collision. Using the conservation of momentum,

$$m_1 v_0 = m_1 v_1 + m_2 v_2 \quad (5)$$
$$m_2 v_2 = m_2 v_2' + m_3 v_3 \quad (6)$$

$$\text{Hence, } m_1 v_0 = m_1 v_1 + m_2 v_2' + m_3 v_3$$
$$\text{From } v_3 - v_2' = v_2, m_1 v_0 = m_1 v_1 + m_2(v_3 - v_2) + m_3 v_3$$
$$\text{Rearranging, } v_3 = \frac{m_1 v_0 - m_1 v_1 + m_2 v_2}{m_2 + m_3}$$

$$\text{Multiplying Equation (3) with } m_1 : m_1 v_2 - m_1 v_1 = m_1 v_0 \quad (7)$$

$$\text{Adding Equations (5) and (7): } m_1 v_2 + m_2 v_2 = 2m_1 v_0$$

$$\text{Hence, } v_2 = \frac{2m_1}{m_1 + m_2} v_0 \tag{8}$$

$$\text{Multiplying Equation (3) with } m_2 : m_2 v_2 - m_2 v_1 = m_2 v_0 \tag{9}$$

$$\text{Equation (5) - Equation (9): } (m_1 - m_2) v_0 = (m_1 + m_2) v_1$$

$$\text{Hence, } v_1 = \frac{m_1 - m_2}{m_1 + m_2} v_0 \tag{10}$$

Substituting the results in (8) and (10) into $v_3 = \frac{m_1 v_0 - m_1 v_1 + m_2 v_2}{m_2 + m_3}$:

$$v_3 = \frac{m_1(m_1 + m_2) - m_1(m_1 - m_2) + m_2(2m_1)}{(m_1 + m_2)(m_2 + m_3)} v_0 = \frac{4m_1 m_2}{(m_1 + m_2)(m_2 + m_3)} v_0$$

ii. To obtain the maximum value of v_3, differentiate v_3 w.r.t. m_2.

$$\begin{aligned} \frac{dv_3}{dm_2} &= \frac{(m_1 + m_2)(m_2 + m_3)(4m_1 v_0) - 4m_1 m_2 v_0 \frac{d}{dm_2}(m_1 + m_2)(m_2 + m_3)}{[(m_1 + m_2)(m_2 + m_3)]^2} \\ &= \frac{(m_1 + m_2)(m_2 + m_3)(4m_1 v_0) - 4m_1 m_2 v_0 (m_1 + 2m_2 + m_3)}{[(m_1 + m_2)(m_2 + m_3)]^2} \end{aligned}$$

For the maximum $\frac{dv_3}{dm_2} = 0$. Hence, for the maximum value of v_3,

$$\begin{aligned} (m_1 + m_2)(m_2 + m_3)(4m_1 v_0) &= 4m_1 m_2 v_0 (m_1 + 2m_2 + m_3) \\ m_1 m_2 + m_1 m_3 + m_2^2 + m_2 m_3 &= m_1 m_2 + 2m_2^2 + m_2 m_3 \\ m_1 m_3 &= m_2^2, \text{ and } m_2 = \sqrt{m_1 m_3} \end{aligned}$$