# Question 1

Speed of heavy ball immediately before hitting the ground is (Conservation of Energy):

$$\frac{1}{2}Mv_b^2 = Mgh$$
$$v_b = \sqrt{2gh}$$

**Note 1.** *You may find the speed using kinematics as well.*

Speed of heavy ball immediately after bouncing off the ground is

$$v_a = v_b = \sqrt{2gh}$$

Speed of light ball immediately before colliding with heavy ball is (Conservation of Energy):

$$\frac{1}{2}Mv_l^2 = Mgh$$
$$v_l = \sqrt{2gh}$$

**Note 2.** *You may find the speed using kinematics as well.*

By Conservation of Momentum, and the fact that speed of approach equals speed of separation ($v_a + v_l = u - U$),

$$\begin{aligned} Mv_a - mv_l &= MU + mu \\ &= M(u - v_a - v_l) + mu \\ (M-m)\sqrt{2gh} &= M(u - 2\sqrt{2gh}) + mu \\ (3M-m)\sqrt{2gh} &= (M+m)u \\ u &= \frac{3M-m}{M+m}\sqrt{2gh} \end{aligned}$$

Since $M \gg m$, we have

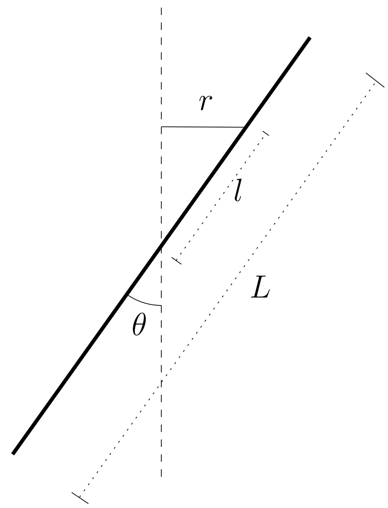
$$u = 3\sqrt{2gh}$$

By Conservation of Energy,

$$\frac{1}{2}mu^2 = mgh'$$
$$h' = \frac{u^2}{2g}$$
$$\boxed{h' = 9h}$$

**Note 3.** *You may find the height using kinematics as well.*

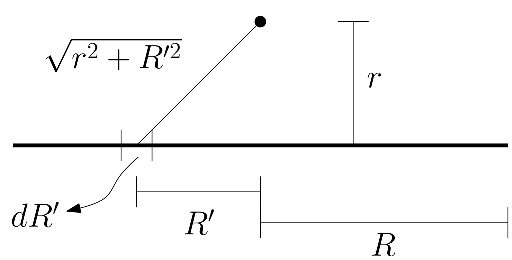
# Question 2




$$
\begin{aligned}
I &= \int r^2 \, \mathrm{d}m \\
&= \int_{-\frac{L}{2}}^{\frac{L}{2}} [l \sin(\theta)]^2 \left(\frac{M}{L} \, \mathrm{d}l\right) \\
&= \frac{M \sin^2(\theta)}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} l^2 \, \mathrm{d}l
\end{aligned}
$$

$$\boxed{I = \frac{ML^2 \sin^2(\theta)}{12}}$$

# Question 3




Mass of sphere is concentrated where the dot is.

By invoking the Shell Theorem, we can assume that the mass of the solid sphere is concentrated at its center, which is at a vertical distance $r$ away from the center of the disk. The potential energy associated with the sphere and a thin ring of radius $R'$ is given by

$$
\begin{aligned}
\mathrm{d}U &= -\frac{GM \, \mathrm{d}M}{\sqrt{r^2 + R'^2}} \\
&= -\frac{GM}{\sqrt{r^2 + R'^2}} \cdot \frac{M}{\pi R^2} \, 2\pi R' \, \mathrm{d}R' \\
&= -\frac{2GM^2}{R^2} \cdot \frac{R'}{\sqrt{r^2 + R'^2}} \, \mathrm{d}R'
\end{aligned}
$$

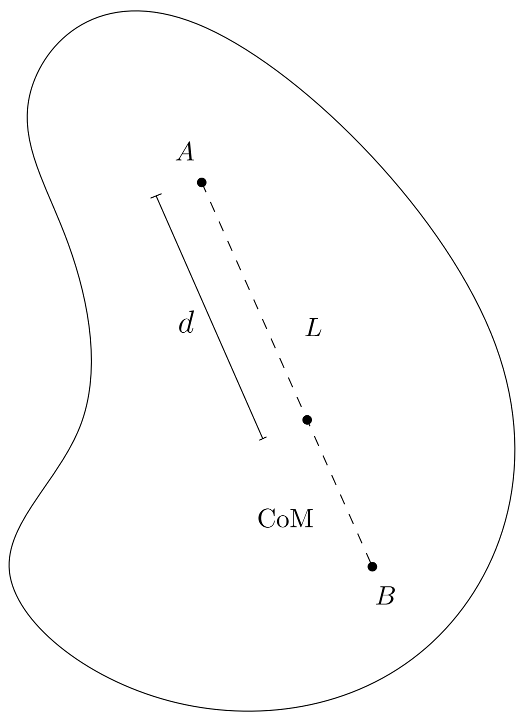
The total potential energy is then

$$\begin{aligned} U &= \int_0^R -\frac{2GM^2}{R^2} \cdot \frac{R'}{\sqrt{r^2+R'^2}}\,\mathrm{d}R' \\ &= -\frac{2GM^2}{R^2}\int_0^R \frac{R'}{\sqrt{r^2+R'^2}}\,\mathrm{d}R' \end{aligned}$$

$$\boxed{U = -\frac{2GM^2}{R^2}\left(\sqrt{r^2+R^2}-r\right)}$$

## Question 4

Yes, he can! This is because ...




Defining $d$ to be the distance from the first pivot to the center of mass, $m$ to be the mass of the pendulum, $I_{\text{COM}}$ to be the moment of inertia of the pendulum about its center of mass,

$$\text{First pivot} \qquad \omega^2 = \frac{mgd}{I_{\text{COM}} + md^2} \tag{1}$$

$$\text{Second pivot} \qquad \omega^2 = \frac{mg(L-d)}{I_{\text{COM}} + m(L-d)^2} \tag{2}$$

**Note 4.** *The above came from the equation* $\omega^2 = \frac{mgd}{I}$.

We equate the equations above by eliminating $I_{\text{COM}}$ to get,

$$\begin{aligned}
\frac{mgd}{\omega^2} - md^2 &= \frac{mg(L-d)}{\omega^2} - m(L-d)^2 \\
(L-d)^2 - d^2 &= \frac{g}{\omega^2}(L-d-d) \\
L(L-2d) &= \frac{g}{\omega^2}(L-2d) \\
g &= L\omega^2 \\
&= \frac{4\pi^2 L}{T^2}
\end{aligned}$$

$\ldots g$ depends only on $L$ and $T$! There is no dependence on the moment of inertia!

# Question 5

The general doppler effect equation for sound is

$$f = f_o \frac{v \pm v_o}{v \mp v_s}$$

where $f$ is the shifted frequency, $f_o$ is the original frequency, $v_o$ and $v_s$ are the speeds of the observer and source respectively.

Using the above, and defining $v$ as the speed of sound in air, we find the perceived frequency by the insect to be

$$f_{\text{ins}} = f_{\text{bat}} \frac{v + v_{\text{ins}}}{v - v_{\text{bat}}}$$

The frequency of the sound reflected from the insect in the insect's frame is also given by the above. The frequency of the reflected sound in the bat's frame is

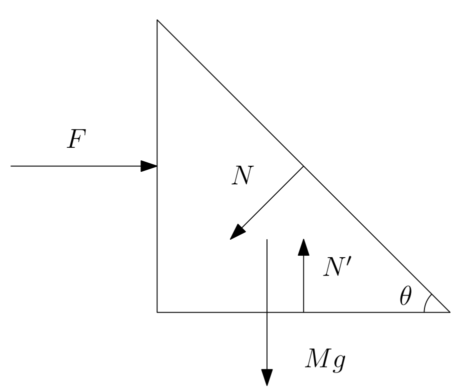
$$f_{\text{refl}} = f_{\text{bat}} \frac{v + v_{\text{ins}}}{v - v_{\text{bat}}} \frac{v + v_{\text{bat}}}{v - v_{\text{ins}}}$$

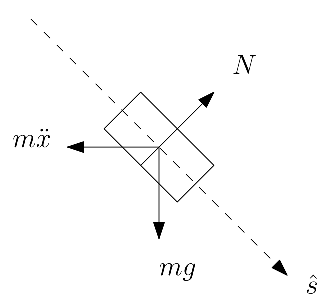
The rest is just trivial Mathematics to show that

$$\boxed{v_{\text{ins}} = v \frac{f_{\text{refl}}(v - v_{\text{bat}}) - f_{\text{bat}}(v + v_{\text{bat}})}{f_{\text{refl}}(v - v_{\text{bat}}) + f_{\text{refl}}(v + v_{\text{bat}})}}$$

**Note 5.** *A simple way to check the answer is to take $f_{refl} = f_{bat}$. This is when the insect is moving at the same speed as the bat, i.e. $v_{ins} = -v_{bat}$. The minus sign is due to the direction involved — we take positive for $v_{ins}$ to be towards the bat, which is in the direction opposite to the direction which the bat is travelling in. Putting the equality of the frequencies into the answer gives the expected result.*

# Question 6



In the accelerated frame of the wedge, with acceleration at $\ddot{x}$, we have the following forces (and resultant force) on the block (the forces are resolved along the direction of the slope ($\hat{\mathbf{s}}$) and the axis perpendicular to it):

$$m\ddot{x}\sin(\theta) + mg\cos(\theta) = N \tag{3a}$$
$$mg\sin(\theta) - m\ddot{x}\cos(\theta) = \ddot{s} \tag{3b}$$

The horizontal forces on the wedge in the inertial frame are then related by:

$$F - N\sin(\theta) = M\ddot{x} \tag{4}$$

We equate all the above while eliminating $\ddot{x}$ and $N$ to get

$$\ddot{s} = g\sin(\theta) - \frac{F - mg\sin(\theta)\cos(\theta)}{M + m\sin^2(\theta)}\cos(\theta)$$

where in the process we have

$$\ddot{x} = \frac{F - mg\sin(\theta)\cos(\theta)}{M + m\sin^2(\theta)}$$

The acceleration of the block back in the inertial frame is then

$$\boxed{a_x = \ddot{s}\cos(\theta) + \ddot{x} = g\sin(\theta)\cos(\theta) + \frac{F - mg\sin(\theta)\cos(\theta)}{M + m\sin^2(\theta)}\sin^2(\theta)}$$

$$\boxed{a_y = \ddot{s}\sin(\theta) = \left(g\sin(\theta) - \frac{F - mg\sin(\theta)\cos(\theta)}{M + m\sin^2(\theta)}\cos(\theta)\right)\sin(\theta)}$$

# Question 7

## Part A

Downward force due to diver:

$$\begin{aligned} F_{\text{diver}} &= mg - U \\ &= 120g - \rho_{\text{water}} V g \\ &\approx 392\,\text{N} \end{aligned}$$

Downward force due to cable:

$$\begin{aligned} F_{\text{rope}}(x) &= mg - U \\ &= \mu x g - \rho_{\text{water}} x \pi \frac{d^2}{4} g \\ &\approx 7.701x\,\text{N} \end{aligned}$$

Tension at $x$ is therefore

$$\boxed{T = (392 + 7.70x)\,\text{N}}$$

### Part B

$$
\begin{aligned}
v &= \sqrt{\frac{T}{\mu}} \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= \sqrt{\frac{392 + 7.70x}{1.10}} \\
\int_0^t \mathrm{d}t &= \int_0^{100} \sqrt{\frac{1.10}{392 + 7.70x}}\,\mathrm{d}x
\end{aligned}
$$

$$\boxed{t = 3.89\,\mathrm{s}}$$

## Question 8

### Part A

$$\boxed{\text{In the direction of motion.}}$$

### Part B

The length of the major-axis is:

$$a = \frac{d_{\text{Sun-Earth}} + d_{\text{Sun-Mars}}}{2} = 1.89 \times 10^{11}\,\mathrm{m}$$

Kepler's Third Law gives

$$T = \sqrt{\frac{4\pi^2 a^3}{GM}} = 4.481 \times 10^7\,\mathrm{s}$$

Time taken is half the period, thus giving

$$\boxed{t = \frac{4.481 \times 10^7}{2}\,\mathrm{s} = 259\,\mathrm{days}}$$

### Part C

By Conservation of Angular Momentum, and defining subscripts $a$ and $p$ to be the aphelion and perihelion respectively,

$$
\begin{aligned}
m v_a r_a &= m v_p r_p \\
v_a &= v_p \frac{r_p}{r_a} \\
&= v_p \frac{a - \varepsilon a}{a + \varepsilon a} \\
&= v_p \frac{1 - \varepsilon}{1 + \varepsilon}
\end{aligned}
$$

By Conservation of Energy, defining $M_s$ to be the mass of the Sun and $\varepsilon$ to be the eccentricity, and using the above result,

$$\begin{aligned}
\frac{1}{2}mv_a^2 - \frac{GM_s m}{r_a} &= \frac{1}{2}mv_p^2 - \frac{GM_s m}{r_p} \\
\frac{1}{2}v_a^2 - \frac{GM_s}{a(1+\varepsilon)} &= \frac{1}{2}v_p^2 - \frac{GM_s}{a(1-\varepsilon)} \\
\left(v_p \frac{1-\varepsilon}{1+\varepsilon}\right)^2 - \frac{2GM_s}{a(1+\varepsilon)} &= v_p^2 - \frac{2GM_s}{a(1-\varepsilon)} \\
v_p^2 \frac{4\varepsilon}{(1+\epsilon)^2} &= \frac{4GM_s}{a}\frac{\varepsilon}{1-\varepsilon^2} \\
v_p^2 &= \frac{GM_s}{a}\frac{1+\varepsilon}{1-\varepsilon}
\end{aligned}$$

The eccentricity can be found easily by:

$$\begin{aligned}
\varepsilon &= \frac{c}{a} \\
&= \frac{a - d_{\text{Sun-Earth}}}{a} \\
&= 0.26
\end{aligned}$$

The speed when it just leaves Earth is therefore:

$$\boxed{v_p = \sqrt{\frac{GM_s}{a}\frac{1+\varepsilon}{1-\varepsilon}} \approx 3.46 \times 10^4 \,\mathrm{m\,s^{-1}}}$$

## Part D

The Earth is moving through space at a speed of:

$$v_e = \omega r = \frac{2\pi}{T} r = 2.989 \times 10^4 \,\mathrm{m\,s^{-1}}$$

The spacecraft has to escape Earth's gravity, and defining $M_e$ to be the mass of Earth, hence we have:

$$\begin{aligned}
\frac{1}{2}m(v_e + v)^2 - \frac{GM_e m}{R_e} &= \frac{1}{2}mv_p^2 \\
(v_e + v)^2 &= v_p^2 + \frac{2GM_e}{R_e} \\
v &= \sqrt{v_p^2 + \frac{2GM_e}{R_e}} - v_e
\end{aligned}$$

$$\boxed{v = 6470 \,\mathrm{m\,s^{-1}}}$$

# PC1141 09/10 corrections

January 8, 2013

## 1 Q4

Note: The aforementioned method of determining g will only work when $L \neq 2d$. For a system of physical pendulum (regardless of its uniformity), two points from the same side of the center of mass will share the same period. This is evident from the proving shown:

$$\begin{aligned} T &= 2\pi\sqrt{\frac{I}{mgh}} \\ \frac{T^2}{4\pi^2} &= \frac{I_{cm} + mh^2}{mgh} \end{aligned} \tag{1}$$

As shown, the equation is a quadratic one and thus, has 2 solutions for h. Therefore, in order to determine the value of g using the method as mentioned in the quesion, the boy should take caution as not to measure T from the point where the value of h is the same as that measured from the other side.

## 2 Q8(c)

$\epsilon = 0.206$ and not 0.26