Question 1

Mass removed from the pizza $= \frac{(\frac{R}{4})^2\pi}{R^2\pi}M = \frac{M}{16}$

From the centre of the circle,

$$\boxed{\therefore x'_{cm} = \frac{(-\frac{M}{16})(-\frac{3R}{4})}{\frac{15}{16}M} = \frac{R}{20}}$$

$$\boxed{\therefore y'_{cm} = 0}$$

Question 2

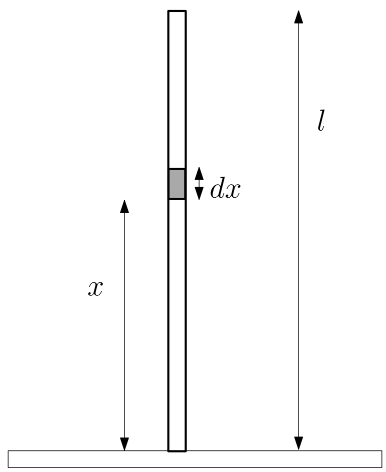


Figure 1: Question 2 sketch

Consider a part of the chain dx from a height of x as shown in Figure 1. This part of the chain has a mass of dm which is

$$dm = \frac{dx}{L}M$$

By the conservation of energy, the velocity of the chain when it reaches the weighing scale is

$$v = \sqrt{2gx} = \frac{dx}{dt}$$

Rearranging,

$$dt = \frac{1}{\sqrt{2gx}}dx \tag{1}$$

The impulse, or change in momentum when this part of the chain hits the weighing scale is given by

$$dp = v\,dm = \frac{M}{L}dx\sqrt{2gx} \tag{2}$$

Putting together Equations 1 and 2, we have the force exerted on the weighing scale due to the dropping of the chain by a length x,

$$F_{drop} = \frac{dp}{dt} = 2Mg\frac{x}{L}$$

The reading on the scale is the combination of the weight of the chain on the scale and the force exerted due to the drop,

$$\boxed{\therefore F_{total} = F_{drop} + Mg\frac{x}{L} = 3Mg\frac{x}{L}}$$

Question 3

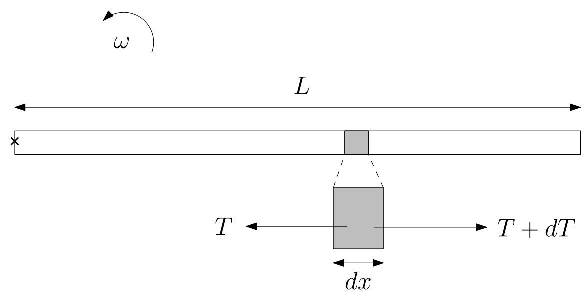


Figure 2: Question 3 sketch

Consider a portion of rope with length dx from the axis of rotation as shown in Figure 2. From the equations for circular motion

$$\begin{aligned} T - (T + dT) &= dm\, x\omega^2 \\ -dT &= \frac{dx}{L} Mx\omega^2 \end{aligned} \tag{3}$$

We can then proceed to integrate the function. To set-up the boundary conditions, we note that at $x = L$, dT is zero (it has nothing attached to the end). At a length r from the centre, dT is equal to the centripetal force acting on that point. Therefore,

$$\begin{aligned} \int_{T(r)}^{0} -dT &= \frac{M}{L}\omega^2 \int_r^L x dx \\ T(r) - 0 &= \frac{M}{L}\omega^2 \left(\frac{L^2}{2} - \frac{r^2}{2}\right) \end{aligned}$$

$$\boxed{\therefore T(r) = \frac{M\omega^2}{2L}(L^2 - r^2)}$$

Question 4

Using the conservation of energy, we find the velocity of the water flowing out at the bottom of the tank

$$\begin{aligned} \rho g x &= \frac{1}{2}\rho v^2 \\ v &= \sqrt{2gx} \end{aligned} \tag{4}$$

The rate of volume flow is then equal to va. We can also find the rate of volume flow by finding the change in height of the water in the tank. (Note that the height is decreasing and $\frac{dx}{dt}$ is negative.) Therefore,

$$\begin{aligned} \frac{dV}{dt} &= a\sqrt{2gx} = -A\frac{dx}{dt} \\ dt &= -\frac{A}{a}\frac{1}{\sqrt{2gx}}\, dx \end{aligned} \tag{5}$$

We want to find the time taken t when the water level changes from H to 0,

$$\int_0^t dt = \int_H^0 -\frac{A}{a}\frac{1}{\sqrt{2gx}}\,dx$$

$$\boxed{\therefore t = \frac{A}{a}\sqrt{\frac{2H}{g}}}$$

Question 5

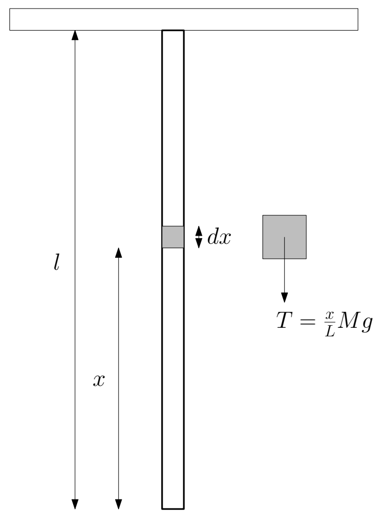


Figure 3: Question 5 sketch

Consider a part of the cable of length dx a distance x from the bottom of the cable shown in Figure 3.

$$\begin{aligned} v = \frac{dx}{dt} &= \sqrt{\frac{T}{\mu}} \\ &= \sqrt{\frac{xMg}{L}\frac{L}{M}} \\ &= \sqrt{xg} \end{aligned}$$

Integrating with respect to time, we have the time taken for the displacement to reach the top of the cable

$$\int_0^t dt = \int_0^L \frac{1}{\sqrt{xg}}dx$$

$$\boxed{\therefore t = 2\sqrt{\frac{L}{g}}}$$

Question 6

Part A

From the orbital velocity V and radius R of the planet and the condition for circular motion,

$$F = \frac{GMm}{R^2} = \frac{mV^2}{R} \tag{6}$$

Using the conservation of energy,

$$\frac{1}{2}mv'^2 - \frac{GMm}{R} = 0$$
$$\frac{1}{2}mv'^2 - mV^2 = 0$$
$$v'^2 = 2V^2$$

$$\boxed{\therefore \quad \text{Escape velocity, } v' = \sqrt{2}V}$$

Note that as long as the rock is launched above the horizontal, the direction of launch does not matter because it would have already escaped from the planet.

Part B

By Gauss' Law, the amount of gravitational force acting on an object placed *inside* a spherical object of mass M and a distance h from its centre is proportional to the mass enclosed by a spherical surface of radius h. Therefore, after falling to a height h from the centre of the planet,

$$F = -\frac{Gm\left(\frac{4}{3}\pi h^3\rho\right)}{h^2} = -\frac{4}{3}Gm\pi\rho h \tag{7}$$

This equation is similar to a simple harmonic motion ($F = -kx$). If we assume no energy is lost due to dissipative forces and the object is not destroyed, the object will then return to its original location.

Its angular frequency ω is therefore

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{\frac{4}{3}Gm\pi\rho}{m}} = \sqrt{\frac{4}{3}G\pi\rho} \tag{8}$$

From Equation 6 we can calculate the density ρ,

$$M = \frac{RV^2}{G} = \frac{4}{3}\pi R^3\rho$$
$$\rho = \frac{3}{4}\frac{V^2}{R^2G\pi}$$

and therefore calculate the period t

$$\boxed{\therefore t = \frac{2\pi}{\omega} = \frac{2\pi R}{V}}$$

Question 7

Part A

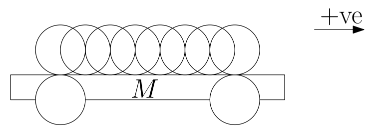
The relative velocity of the N men, v, with respect to the cart is

$$v = v'_m + v'_c \tag{9}$$

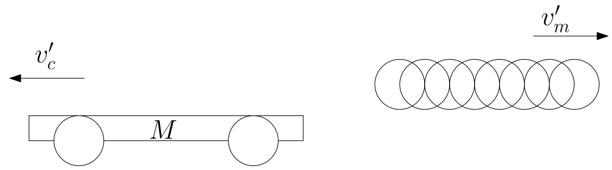
By conservation of momentum, we have

$$v'_cM - Nmv'_m = 0$$
$$v'_cM - Nmv'_c + Nmv = 0$$

$$\boxed{v'_c = \frac{Nm}{M+Nm}v}$$

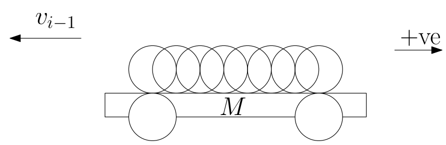


(a) Initial state (at rest)

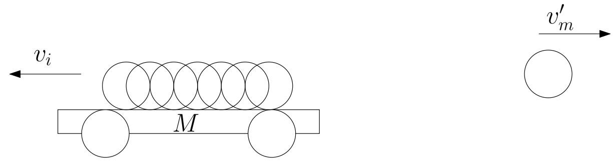


(b) Final state (after jumping out)

Figure 4: Question 7a sketch



(a) Initial state, $[N-(i-1)]$ men



(b) Final state, $(N-i)$ men

Figure 5: Question 7b sketch

Part B

Now suppose after $i-1$ men have jumped off the cart, the cart is moving with a velocity of v_c. The ith man then jumps off with a velocity of v with respect to the cart, and the final velocities of the cart and ith man (w.r.t ground) are v_i and v'_m respectively.

The relative velocity of the ith man, v, with respect to the cart v_i is

$$v = v'_m - v_i \tag{10}$$

By conservation of momentum,

$$\begin{aligned} v_{i-1}[M+(N-i+1)m] &= v_i[M+(N-i)m] - mv'_m \\ &= v_i[M+(N-i)m] - m(v_i+v) \\ &= v_i[M+(N-i+1)m] - mv \\ \therefore v_i &= v_{i-1} + \frac{mv}{M+(N-i+1)m} \end{aligned} \tag{11}$$

This is the case for the ith man. To find the final velocity of the cart after N men have jumped off, we take the summation of Equation 11, which gives

$$v_N - v_0 = \sum_{i=1}^{N} \frac{mv}{M+(N-i+1)m}$$

$$\boxed{\therefore v_N = \sum_{i=1}^{N} \frac{mv}{M + (N - i + 1)m}}$$

Part C

Comparing the two results we have obtained, it may be observed that the case in Part B results in a cart of higher speed. This can be seen by writing the answer to Part A as

$$v_c' = \frac{Nm}{M + Nm} v = \sum_{i=1}^{N} \frac{mv}{M + Nm}$$

Each term in the summation of v_c' is smaller than its counterpart in v_N, hence v_c' must be smaller than v_N.

Note 1. *This is because it takes more work for one man to jump off the cart with a speed v relative to the cart as compared to N men who collectively jump off at a relative speed of v.*

Question 8

Part A

First, we calculate the time interval between the 1st and 2nd impact using the kinematics equation

$$\alpha h = 0 + \frac{1}{2} g \left(\frac{t}{2}\right)^2$$

$$\implies \quad t = 2\sqrt{\frac{2\alpha h}{g}} \tag{12}$$

We now analyse the collision. First, we take the impact time to be Δt. The horizontal impulse by friction due to the rotation of the ball is

$$f = \frac{m\Delta v_x}{\Delta t}$$

$$\mu N = \frac{m(v_x - 0)}{\Delta t}$$

$$\implies \quad v_x = \frac{\mu N \Delta t}{m} \tag{13}$$

The vertical impulse due to the normal contact force is therefore

$$N = \frac{m\Delta v_y}{\Delta t}$$

$$= \frac{m(\sqrt{2g\alpha h} + \sqrt{2gh})}{\Delta t}$$

$$\implies \quad N\Delta t = m\sqrt{2gh}(\sqrt{\alpha} + 1) \tag{14}$$

Note 2. *Note that this is not the weight of the ball.*

We can then substitute Equation 14 into Equation 13 to obtain

$$v_x = \mu\sqrt{2gh}(\sqrt{\alpha} + 1) \tag{15}$$

The distance between the two points of impact is then

$$\boxed{v_x t = 4\mu h(\alpha + \sqrt{\alpha})}$$

Part B

We know that friction causes the angular velocity to decrease during the impact. For the ball to continuously slip, the final angular velocity must be at least equal to (rebound with no spin) or greater than zero (still spinning) after the impact .

We can find the angular acceleration from

$$\begin{aligned}\tau = I\dot{\omega} &= -Rf \\ -\dot{\omega} &= \frac{Rf}{I}\end{aligned}$$

Then,

$$\begin{aligned}-\Delta\omega = \omega_i - \omega_f &= -\dot{\omega}\Delta t \\ &= \frac{R\mu N\Delta t}{I} \\ &= \frac{R\mu m\sqrt{2gh}(\sqrt{\alpha}+1)}{\frac{2}{5}mR^2} \\ \omega_f &= \omega_i - \frac{5\mu\sqrt{2gh}(\sqrt{\alpha}+1)}{2R} \end{aligned} \tag{16}$$

$$\text{If } \omega_f > 0, \text{then } \omega_i > \frac{5\mu\sqrt{2gh}(\sqrt{\alpha}+1)}{2R}$$

$$\boxed{\therefore \omega_{min} = \frac{5\mu\sqrt{2gh}(\sqrt{\alpha}+1)}{2R}}$$

Part C

If the ball stops slipping before the impact ends, then the friction will just cause the ball to 'roll' during impact. Then by conservation of energy, choosing the points where the ball is at its maximum height,

$$mgh + \frac{1}{2}I\omega^2 = mg\alpha h + \frac{1}{2}mv_s^2 + \frac{1}{2}I\omega_f^2 \tag{17}$$

where v_s is the horizontal velocity of the ball. The angular velocity is then related by the no-slip condition $v_s = r\omega$.

$$\begin{aligned}\frac{1}{2}mv_s^2 + \frac{1}{2}\frac{2mR^2}{5}\frac{v_s^2}{R^2} &= mgh(1-\alpha) + \frac{1}{5}mR^2\omega_0^2 \\ v_s^2 &= \frac{10}{7}\left(gh(1-\alpha) + \frac{1}{5}R^2\omega_0^2\right)\end{aligned} \tag{18}$$

$$\begin{aligned}\implies \quad s &= v_s t \\ &= \sqrt{\frac{10}{7}\left(gh(1-\alpha) + \frac{1}{5}R^2\omega_0^2\right)} \times 2\sqrt{\frac{2\alpha h}{g}}\end{aligned} \tag{19}$$

$$\boxed{\therefore s = 4\sqrt{\left(\frac{5}{7}\alpha h^2(1-\alpha) + \frac{1}{7}\frac{\alpha h R^2\omega_0^2}{g}\right)}}$$

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8(b) For the ball to stop slipping during collision,

$$v_x = R\omega$$

The minimum ω_o for the ball to slip throughout the impact is the ω_o which will result in the condition above satisfied just before the collision ends.

$$fR = I\alpha$$
$$\alpha_{angular} = \frac{5\mu g}{2R}$$
$$v_x = at$$
$$a = \mu g$$
$$t = \frac{v_x}{\mu g}$$

Substituting all the quantities into the equation $\omega = \omega_o - \alpha_{angular}\, t$, with v_x as in the answer from part (a),

$$\omega_o = \frac{7v_x}{2R}$$
$$\omega_o = \frac{7\mu\sqrt{2gh}\left(\sqrt{\alpha}+1\right)}{2R}$$

(c) The condition stated in part (b) is again needs to be satisfied. Using the same quantities computed in part (b),

$$v_x = \frac{2R\omega_o}{7}$$

The time taken is as computed in part (a),

$$\therefore\ x = \frac{4R\omega_o}{7}\sqrt{\frac{2\alpha h}{g}}$$