Question 1:

(a) $$V = k\int \frac{dq}{r} = \frac{k}{R}\int dq = \frac{kQ}{R}$$

(b) The linear charge density is given by the function $\lambda(\theta) = \alpha\theta$, where α is some constant (to be determined) and $0 < \theta < 2\pi$ is the angle measured counter-clockwise from the positive x-axis.

An element Δq located at $\theta = \theta_0$ subtending an angular width of $\Delta\theta$ has a charge of

$$\Delta q = \lambda R\Delta\theta = \alpha\theta_0 R\Delta\theta$$

Consider the case where $\theta_0 < \pi$. There would be an element $\Delta q'$ on the opposite side of the ring (i.e. $\theta = \theta_0 + \pi$) with a charge of

$$\Delta q' = \alpha(\theta_0 + \pi)R\Delta\theta$$

The E field contributions from Δq and $\Delta q'$ are parallel and opposite in direction. They cancel out partially and since they are equidistant from the center, a test charge feels only an E field due to an effective charge of $\Delta q' - \Delta q = \alpha\pi R\Delta\theta$ from the lower half of the loop.

Note that this effective charge does not depend on the choice of θ_0. Therefore, the E field at the center of this loop is exactly the same as the E field due to a semicircular arc of the same radius, on the lower half of the plane, with a uniform linear charge density $\alpha\pi$.

Moreover, note that $\alpha\pi$ is the median (and thus the mean value since λ varies linearly with θ) of λ over the interval $(0, 2\pi)$. Therefore, $\alpha\pi = \frac{Q}{2\pi R}$.

It is much easier to evaluate the E field using the semicircular arc. By symmetry, the E field in the x and z directions is zero. Using the same coordinate θ, a charge dq located at θ has cartesian coordinates

$$x = R\cos\theta$$
$$y = R\sin\theta$$

The amonut of E field in the y-direction that dq contributes is given by

$$dE_y = \frac{kdq}{R^3}(0 - y)$$

$$\begin{aligned} dE_y &= \frac{k(\alpha\pi R d\theta)}{R^3}(-R\sin\theta) \\ &= -\frac{k(\frac{Q}{2\pi R}Rd\theta)}{R^2}\sin\theta \\ &= -\frac{kQ}{2\pi R^2}\sin\theta d\theta \end{aligned}$$

The total E field is obtained by integrating over the range of θ subtended by the semicircle.

$$\begin{aligned} E_y &= \int_{\pi}^{2\pi} -\frac{kQ}{2\pi R^2}\sin\theta d\theta \\ &= \frac{kQ}{2\pi R^2}[\cos\theta]_{\pi}^{2\pi} \\ &= \frac{kQ}{\pi R^2} \end{aligned}$$

The E field at the origin is $\boldsymbol{E} = \frac{kQ}{\pi R^2}\hat{\boldsymbol{y}}$.

Question 2:

(a) The capacitor is discharging through a resistor, a typical RC circuit.

To calculate the resistance and capacitance of the sphere, divide the sphere into n thin concentric shells of equal thickness $\Delta r = \frac{b-a}{n}$ each. When n is large, the i^{th} shell has an area of approximately $4\pi r_i^2$, where r_i is its radius.

For the i^{th} shell, its capacitance C_i is given by

$$C_i = \frac{\kappa\epsilon_0(4\pi r_i^2)}{\Delta r}$$

The shells are in series. Hence, the total capacitance is

$$\begin{aligned} C &= \left(\sum_{\forall i}\frac{1}{C_i}\right)^{-1} \\ &= \left(\int_a^b \frac{dr}{4\pi\kappa\epsilon_0 r^2}\right)^{-1} \\ &= 4\pi\kappa\epsilon_0\left(\frac{1}{a}-\frac{1}{b}\right)^{-1} \end{aligned}$$

For the i^{th} shell, its resistance R_i is given by

$$R_i = \frac{\Delta r}{\sigma(4\pi r_i^2)}$$

The shells are in series. Hence, the total resistance is

$$\begin{aligned} R &= \sum_{\forall i} R_i \\ &= \int_a^b \frac{dr}{4\sigma\pi r^2} \\ &= \frac{1}{4\sigma\pi}\left(\frac{1}{a} - \frac{1}{b}\right) \end{aligned}$$

Let the charge of the inner sphere be $q(t)$. The potential difference across the resistor is equal to the potential difference between the capacitor. Thus,

$$\frac{q}{C} = -R\frac{dq}{dt}$$

The solution of this first-order ODE is

$$q = q_0 \exp\left(-\frac{t}{RC}\right)$$

where q_0 is the charge at $t = 0$, which is given by the initial condition

$$\begin{aligned} q_0 &= CV_0 \\ &= 4\pi\kappa\epsilon_0 V_0 \left(\frac{1}{a} - \frac{1}{b}\right)^{-1} \end{aligned}$$

and also observe that

$$RC = \frac{\kappa\epsilon_0}{\sigma}$$

Therefore, $q = \boxed{4\pi\kappa\epsilon_0 V_0 \left(\frac{1}{a} - \frac{1}{b}\right)^{-1} \exp\left(-\frac{\sigma t}{\kappa\epsilon_0}\right)}$.

(b) Taking only the magnitude of the current,

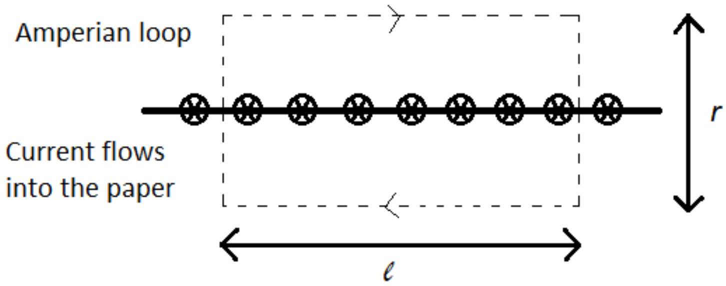
$$\begin{aligned} |i| &= -\frac{dq}{dt} \\ &= \boxed{4\pi V_0 \sigma \left(\frac{1}{a} - \frac{1}{b}\right)^{-1} \exp\left(-\frac{\sigma t}{\kappa\epsilon_0}\right)} \end{aligned}$$

Question 3:

(a) The B field created by an infinite current carrying sheet can be easily determined using Ampere's Law. By symmetry, the B field at any point is normal to the current density (right-hand rule) and depends only on the distance away from the sheet.

Considering only a single infinite current carrying sheet, construct a rectangular Amperian loop that is normal to the current density, with two sides parallel to the sheet (length l) and two sides normal to the plane (breadth r).

In order to make use of the symmetry, the loop should protrude from both sides of the sheet equally by $r/2$. Refer to the diagram.



Along the length of the loop, the B field is parallel to the path of the loop. Along the breadth of the loop, the B field is orthorgonal to the path of the loop. Let the current per unit length be λ. The current enclosed is $l\lambda$. Using Ampere's Law,

$$\oint \boldsymbol{B} \cdot d\boldsymbol{r} = \mu_0 I_{\text{encl}}$$
$$B(2l) = \mu_0(l\lambda)$$
$$B = \frac{\mu_0 \lambda}{2}$$

The B field above/below the sheet is a uniform field that only depends on λ. For the electron not to deviate from its path, it must experience a zero net B field. This happens when a current of (8.0 A/m) flows to the right through the bottom sheet.

(b) From part (a), the current from the top sheet will "neutralize" 8 A/m of the current in the bottom sheet, leaving only an effective current of 2 A/m. The B field is given by

$$B = \frac{\mu_0 \lambda}{2} = \frac{1}{2}(4\pi \times 10^{-7}\ \text{Tm/A})(2\ \text{A/m}) = 1.3 \times 10^{-6}\ \text{T}$$

The direction of the B field is given by the right-hand rule and points out of the paper.

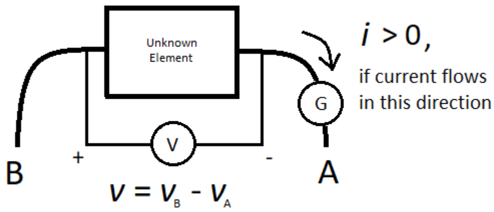
The magnetic force is

$$\boldsymbol{F} = (-e)\boldsymbol{v} \times \boldsymbol{B} = -e(v\hat{\boldsymbol{x}}) \times (B\hat{\boldsymbol{z}}) = evB\hat{\boldsymbol{y}} = (5.0 \times 10^{-19}\ \text{N})\hat{\boldsymbol{y}}$$

The electron deflects towards the top sheet.

Question 4:

Assume that the galvanometer measures positive current as leaving the negative terminal defined by the voltmeter.



Since the current leads voltage, the unknown element should be a capacitor, and not an inductor. Suppose the voltage of the capacitor is given by $v = V_0 \sin(\omega t)$. The current is given by

$$\begin{aligned} i &= \frac{dq}{dt} \\ &= C\frac{dv}{dt} \\ &= \omega C V_0 \cos(\omega t) \\ &= I_0 \cos(\omega t) \end{aligned}$$

The capacitive reactance is given by the ratio of the amplitude of the voltage to current

$$\begin{aligned} \frac{V_0}{I_0} &= \frac{1}{\omega C} \\ &= \frac{1.00\ \text{V}}{1.00\ \text{mA}} \\ &= 1.00\ \text{k}\Omega \end{aligned}$$

The period is 2 s and thus $\omega = 3.14\ \text{s}^{-1}$ and $C = 0.32\ \text{mF}$.

Note: If the opposite convention (positive current defined as one leaving the positive terminal) is assumed, the object will be found to be an inductor of $L = 318$ H, which is not available as one of the choices.

Question 5:

(a)

$$\boldsymbol{E}(0,y,z) = \frac{k(-q)}{(a^2+y^2+z^2)^{\frac{3}{2}}}\begin{pmatrix} a \\ y \\ z \end{pmatrix} + \frac{kq}{(a^2+y^2+z^2)^{\frac{3}{2}}}\begin{pmatrix} -a \\ y \\ z \end{pmatrix}$$

$$= \frac{-2kqa}{(a^2+y^2+z^2)^{\frac{3}{2}}}\hat{\boldsymbol{x}}$$

Using cylindrical coordinates, let $y = r\cos\theta$ and $z = r\sin\theta$.

$$\boldsymbol{E} = \frac{-2kqa}{(a^2+r^2)^{\frac{3}{2}}}\hat{\boldsymbol{x}}$$

Define flux passing from the right to left as positive. For a small ring element,

$$d\boldsymbol{A} = 2\pi r dr(-\hat{\boldsymbol{x}})$$

$$\boldsymbol{E}\cdot d\boldsymbol{A} = \frac{4\pi k q_0 a r}{(a^2+r^2)^{\frac{3}{2}}}dr$$

$$= \frac{qa}{\epsilon_0}\frac{r}{(a^2+r^2)^{\frac{3}{2}}}dr$$

The total electric flux, Φ_E, is

$$\Phi_E = \int \boldsymbol{E}\cdot d\boldsymbol{A}$$

$$= \frac{qa}{\epsilon_0}\int_0^{y_0}\frac{r}{(a^2+r^2)^{\frac{3}{2}}}dr$$

Using the substitution $u = \sqrt{a^2+r^2}$, $\frac{du}{dr} = \frac{r}{u}$,

$$\Phi_E = \frac{qa}{\epsilon_0}\int_a^{\sqrt{a^2+y_0^2}}\frac{u}{u^3}du$$

$$= \frac{qa}{\epsilon_0}\left[-\frac{1}{u}\right]_a^{\sqrt{a^2+y_0^2}}$$

$$= \frac{q}{\epsilon_0}\left(1 - \frac{a}{\sqrt{a^2+y_0^2}}\right)$$

The displacement current, I_D, is given by

$$I_D = \epsilon_0\frac{d\Phi}{dt}$$

$$= \frac{dq}{dt}\left(1 - \frac{a}{\sqrt{a^2+y_0^2}}\right)$$

(b) The conventional current is given by $I_C = -\frac{dq}{dt}$. Check that the sign agrees with the convention for the flux as established in part (a).

Let C be the closed circular loop with radius y_0 in the yz-plane centered at the origin. According to Maxwell-Ampere Law,

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{r} = \mu_0(I_D + I_C)$$

(The line integral is carried in out in a clockwise manner when seen from the $+x$ axis.)

From the symmetry and considering Biot-Savart law, it can be shown that the B field is constant in magnitude and parallel to $d\boldsymbol{r}$ along the entire Amperian loop.

$$\begin{aligned} B(2\pi y_0) &= \mu_0 \left(\frac{dq}{dt} \left(1 - \frac{a}{\sqrt{a^2 + y_0^2}} \right) + \left(-\frac{dq}{dt} \right) \right) \\ &= -\mu_0 \frac{dq}{dt} \frac{a}{\sqrt{a^2 + y_0^2}} \\ B &= \frac{\mu_0}{2\pi} \left(-\frac{dq}{dt} \right) \frac{a}{y_0\sqrt{a^2 + y_0^2}} \qquad \text{(magnitude)} \end{aligned}$$

Using the right hand rule, the B field points in the $-z$ direction at $(0, y_0, 0)$. Thus,

$$\boldsymbol{B}(0, y_0, 0) = \boxed{\frac{\mu_0}{2\pi} \frac{dq}{dt} \frac{a}{y_0\sqrt{a^2 + y_0^2}} \hat{\boldsymbol{z}}}$$

Question 6:

(a) By applying Gauss' law to a cylindrical capacitor holding a linear charge density of λ (on the inner shell) in vacuum, the electric field between the two shells is found to be

$$\boldsymbol{E} = \frac{2k\lambda}{r}\hat{\boldsymbol{r}}$$

Suppose the inner shell has radius a and the outer shell has radius b. The voltage between the cylinders can be found by integrating the electric field along a radial line.

$$\begin{aligned} V_a - V_b &= -\int_b^a \boldsymbol{E} \cdot d\boldsymbol{r} \\ &= \int_a^b \frac{2k\lambda}{r} dr \\ &= 2k\lambda \ln\left(\frac{b}{a}\right) \end{aligned}$$

From the definition of capacitance, the capacitance per unit length is defined as

$$C_L = \frac{\lambda}{V_{ba}} = \frac{1}{2k\ln\left(\frac{b}{a}\right)}$$

The capacitor in this problem can be treated as 2 capacitors connected in parallel. The inner capacitor has $a = R$ and $b = 2R$. The outer capacitor has $a = 2R$ and $b = 3R$. Their equivalent capacitance is

$$C_{L,\text{eq}} = \frac{1}{2k\ln\left(\frac{2R}{R}\right)} + \frac{1}{2k\ln\left(\frac{3R}{2R}\right)}$$
$$= \frac{1}{2k}\left(\frac{1}{\ln 2} + \frac{1}{\ln 3 - \ln 2}\right)$$

(b) Since the 2 cylinders have the same potential difference, the ratio of their charge is the ratio of their capacitance.

$$\frac{\lambda_{\text{inner}}}{\lambda_{\text{outer}}} = \frac{C_{L,\text{inner}}\Delta V}{C_{L,\text{outer}}\Delta V} = \frac{C_{L,\text{inner}}}{C_{L,\text{outer}}} = \frac{\frac{1}{\ln 2}\frac{1}{2k}}{\frac{1}{\ln 3-\ln 2}\frac{1}{2k}} = \frac{\ln 3 - \ln 2}{\ln 2}$$

By conservation of charge,

$$\lambda_{\text{inner}} + (-\lambda) + \lambda_{\text{outer}} = 0$$

Solving the above system of equations gives

$$\lambda_{\text{inner}} = \left(1 - \frac{\ln 2}{\ln 3}\right)\lambda \qquad \text{and} \qquad \lambda_{\text{outer}} = \frac{\ln 2}{\ln 3}\lambda$$

(c) Note that by Gauss's Law, the addition of λ_{extra} to the outer shell has no effect on the E field in the region $r < 3R$. The potential difference between the inner shell and outer shell will still remain zero and hence λ_{extra} will stay on the outer shell.

Question 7:

(a) The B field is calculated using Biot-Savart's Law

$$\boldsymbol{B}(\boldsymbol{r_0}) = \frac{\mu_0 I}{4\pi}\int \frac{d\boldsymbol{r} \times (\boldsymbol{r_0} - \boldsymbol{r})}{|\boldsymbol{r_0} - \boldsymbol{r}|^3}$$

Integrating along one side of the loop, (the variable of integration is y)

$$\boldsymbol{B}(x_0, y_0, z_0) = \frac{\mu_0 I}{4\pi}\int_{-a}^{a} \frac{1}{[(x_0-a)^2 + (y_0-y)^2 + z_0^2]^{\frac{3}{2}}} \begin{pmatrix} 0 \\ dy \\ 0 \end{pmatrix} \times \begin{pmatrix} x_0 - a \\ y_0 - y \\ z_0 \end{pmatrix}$$
$$= \frac{\mu_0 I}{4\pi}\int_{-a}^{a} \frac{dy}{[(x_0-a)^2 + (y_0-y)^2 + z_0^2]^{\frac{3}{2}}} \begin{pmatrix} z_0 \\ 0 \\ -(x_0 - a) \end{pmatrix}$$

(b) Using the substitution, $-y = \sqrt{a^2+z_0^2}\tan\theta$, $-\frac{dy}{d\theta} = \sqrt{a^2+z_0^2}\sec^2\theta$.

$$\begin{aligned}
\boldsymbol{B}(0,0,z_0) &= \frac{\mu_0 I}{4\pi}\int_{\tan^{-1}\frac{a}{\sqrt{a^2+z_0^2}}}^{\tan^{-1}\frac{-a}{\sqrt{a^2+z_0^2}}} \frac{-\sqrt{a^2+z_0^2}\sec^2\theta}{[(a^2+z_0^2)(1+\tan^2\theta)]^{\frac{3}{2}}}d\theta \begin{pmatrix} z_0 \\ 0 \\ a \end{pmatrix} \\
&= \frac{\mu_0 I}{4\pi}\int_{-\tan^{-1}\frac{a}{\sqrt{a^2+z_0^2}}}^{\tan^{-1}\frac{a}{\sqrt{a^2+z_0^2}}} \frac{\cos\theta}{a^2+z_0^2}d\theta \begin{pmatrix} z_0 \\ 0 \\ a \end{pmatrix} \\
&= \frac{\mu_0 I}{4\pi(a^2+z_0^2)}[\sin\theta]_{-\tan^{-1}\frac{a}{\sqrt{a^2+z_0^2}}}^{\tan^{-1}\frac{a}{\sqrt{a^2+z_0^2}}} \begin{pmatrix} z_0 \\ 0 \\ a \end{pmatrix} \\
&= \frac{\mu_0 I}{4\pi(a^2+z_0^2)}\left(\frac{a}{\sqrt{a^2+(a^2+z_0^2)}} - \frac{-a}{\sqrt{a^2+(a^2+z_0^2)}}\right) \begin{pmatrix} z_0 \\ 0 \\ a \end{pmatrix} \\
&= \frac{\mu_0 I a}{2\pi(a^2+z_0^2)\sqrt{2a^2+z_0^2}} \begin{pmatrix} z_0 \\ 0 \\ a \end{pmatrix}
\end{aligned}$$

By symmetry, the x and y components of the B field from opposite sides of the loop cancels. All four sides contribute equally to the z component of the B field. The resulting B field due to the entire loop is

$$\begin{aligned}
\boldsymbol{B} &= 4\left(\frac{\mu_0 I a}{2\pi(a^2+z_0^2)\sqrt{2a^2+z_0^2}}\right)(a\hat{\boldsymbol{z}}) \\
&= \frac{2\mu_0 I a^2}{\pi(a^2+z_0^2)\sqrt{2a^2+z_0^2}}\hat{\boldsymbol{z}}
\end{aligned}$$

(c) When $z_0 \gg a$,

$$\boldsymbol{B} \approx \frac{2\mu_0 I a^2}{\pi z_0^3}\hat{\boldsymbol{z}}$$

The B field along the axis of a magnetic dipole at a large distance away is given by

$$\boldsymbol{B} \approx \frac{\mu_0}{2\pi z_0^3}\boldsymbol{\mu}$$

By comparing coeffients, $\boldsymbol{\mu} = 4a^2 I\hat{\boldsymbol{z}}$. This agrees with the definition of $\boldsymbol{\mu} = I\boldsymbol{A}$.

Question 8:

(a) The magnetic field within a long solenoid is uniform and can be found using Ampere's Law.

$$B = \frac{\mu_0 NI}{L}$$

Consider a concentric circular surface within the solenoid with a radius of $r < R$. The magnetic flux Φ_B through this surface is given by

$$\begin{aligned}\Phi_B &= B(\pi r^2)\\ &= \frac{\mu_0 NI\pi r^2}{L}\end{aligned}$$

From Faraday's Law,

$$\oint \boldsymbol{E} \cdot d\boldsymbol{r} = -\frac{\partial \Phi_B}{\partial t}$$

From symmetry, and with the fact that the induced E field is always tangent to the circle,

$$\begin{aligned}E(2\pi r) &= \frac{\partial}{\partial t}\left(\frac{\mu_0 NI\pi r^2}{L}\right)\\ &= \frac{\mu_0 N\alpha\pi r^2}{L}\\ E &= \frac{\mu_0 N\alpha r}{2L}\end{aligned}$$

Since the current is increasing with time, the induced E field is opposite to the current. Therefore, the Poynting vector $\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B}$ points towards the axis of the solenoid. Its magnitude is given by

$$\begin{aligned}S &= \frac{1}{\mu_0} EB \sin\frac{\pi}{2}\\ &= \frac{1}{\mu_0}\left(\frac{\mu_0 N(\alpha t)}{L}\right)\left(\frac{\mu_0 N\alpha r}{2L}\right)\\ &= \frac{\mu_0 N^2\alpha^2 tr}{2L^2}\end{aligned}$$

(b) The power input is the negative flux of the Poynting vector over the surface of the solenoid (a cylinder). Ignoring the fringe fields at the two ends of the solenoid, the Poynting vector is assumed to be radial throughout the solenoid. The Poynting vector contributes to the flux only at the curved part of the cylinder.

$$\begin{aligned}P_{\text{in}} &= -\oiint \boldsymbol{S} \cdot d\boldsymbol{A}\\ &= \frac{\mu_0 RN^2\alpha^2 t}{2L^2}(2\pi RL)\\ &= \frac{\mu_0 \pi R^2 N^2\alpha^2 t}{L}\end{aligned}$$

(c) The energy stored in the solenoid as a function of time is given by

$$\begin{aligned} U &= \frac{1}{2\mu_0} B^2 (\pi R^2 L) \\ &= \frac{\mu_0 \pi R^2 N^2 I^2}{2L} \\ &= \frac{\mu_0 \pi R^2 N^2 \alpha^2 t^2}{2L} \end{aligned}$$

The power input is given by

$$\begin{aligned} P_{\text{in}} &= \frac{dU}{dt} \\ &= \frac{d}{dt}\left(\frac{\mu_0 \pi R^2 N^2 \alpha^2 t^2}{2L}\right) \\ &= \boxed{\frac{\mu_0 \pi R^2 N^2 \alpha^2 t}{L}} \end{aligned}$$

This agrees with the answer in part (b). The power input is stored as energy in the magnetic field.