Question 1 (9 min 36 sec)

A rod of length L lies perpendicular to an infinitely long, uniform line charge of charge density λ C/m (Figure 1).

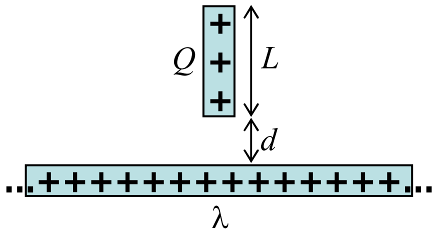


Figure 1

The near end of the rod is a distance d above the line charge. The rod carries a total charge Q uniformly distributed along its length. Find the magnitude of the force that the infinitely long line charge exerts on the rod. [8]

Solution:

Gauss's law,

$$\oint \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0} q_{\text{encl}}$$

$$E \times 2\pi r l = \frac{1}{\varepsilon_0} \lambda l$$

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

$\vec{F} = q\vec{E}$ or Coulomb's law,

$$dF = Edq = E\frac{Q}{L}dr$$

$$F = \frac{\lambda Q}{2\pi\varepsilon_0 L}\int_d^{d+L} \frac{1}{r} dr$$

$$= \frac{\lambda Q}{2\pi\varepsilon_0 L} \ln r \Big|_d^{d+r}$$

$$= \frac{\lambda Q}{2\pi\varepsilon_0 L} \ln\left(1 + \frac{L}{d}\right)$$

Question 2

Consider the circuit in Figure 2,

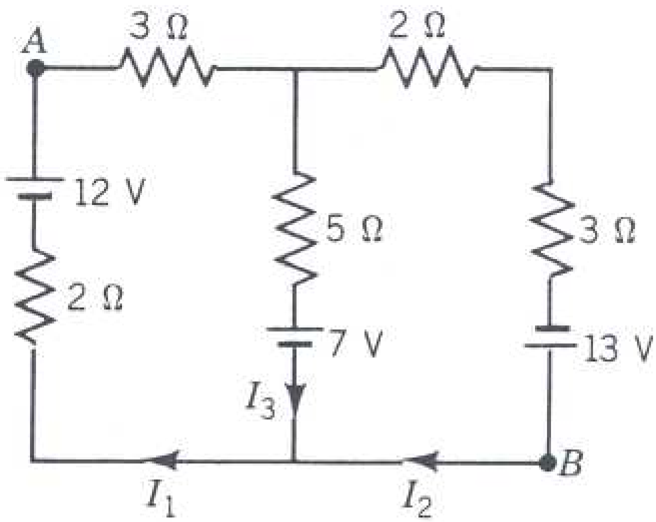


Figure 2

Find

(a) the currents I_1, I_2, and I_3. [6]

(b) the potential difference $V_A - V_B$. [2]

Solution:

Kirchhoff's junction rule,

$$I_2 + I_3 = I_1 \Rightarrow I_1 - I_3 = I_2$$

Kirchhoff's loop rule,

$$-5I_1 + 12 - 5I_3 - 7 = 0 \Rightarrow I_1 + I_3 = 1$$

$$-5I_2 + 20 + 5I_3 \Rightarrow I_2 - I_3 = 4$$

It follows that

$$2I_1 - I_2 = 1$$

$$I_1 + I_2 = 5$$

Therefore,

$$I_1 = 2\text{ A},\ I_2 = 3\text{ A},\ I_3 = -1\text{ A}$$

The potential difference

$$V_A - V_B = 8\text{ V}$$

Question 3

A solenoid consists of a wire of radius a wrapped in a single layer on a paper cylinder of radius r. Show that the time constant is

$$\tau \equiv \frac{L}{R} = \frac{\mu_0 \pi a r}{4\rho}$$

where ρ is the resistivity. [8]

Solution:

Inductance

$$L = \frac{N\Phi_B}{i} = \frac{N}{i}\mu_0 \frac{N}{N \times 2a} i \times \pi r^2 = \frac{\mu_0 N \pi r^2}{2a}$$

Resistance

$$R = \rho\frac{l}{A} = \rho\frac{N \times 2\pi r}{\pi a^2} = \frac{2\rho N r}{a^2}$$

Time constant

$$\tau = \frac{L}{R} = \frac{\mu_0 N \pi r^2}{2a} \times \frac{a^2}{2\rho N r} = \frac{\mu_0 \pi a r}{4\rho}$$

Question 4

In the *R*-*L*-*C* circuit of Figure 3, take $R = 8.00\ \Omega$, $L = 40.0\ \text{mH}$, $C = 20.0\ \mu\text{F}$.

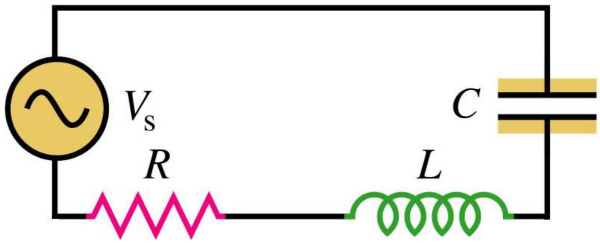


Figure 3

The ac source has voltage amplitude $V = 100\ \text{V}$, and frequency $f = 200/\pi\ \text{Hz}$. Find

(a) the impedance of the circuit. [2]

(b) the peak current in the circuit. [1]

(c) the peak potential difference across *R*, *L*, and *C*. [3]

(d) the peak potential difference across *R* and *L* combined. [2]

Solution:

Impedance of the circuit,

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{64 + \left(400 \times 40 \times 10^{-3} - \frac{1}{400 \times 20 \times 10^{-6}}\right)^2} \approx 109\ \Omega$$

Peak current in the circuit,

$$I = \frac{V}{Z} \approx 0.915\ \text{A}$$

It follows that

$$V_R = RI \approx 7.32\ \text{V}$$

$$V_L = X_L I = \omega L I \approx 14.6\ \text{V}$$

$$V_C = X_C I = \frac{1}{\omega C} I \approx 114\ \text{V}$$

The peak potential difference across *R* and *L* combined

$$\sqrt{V_R^2 + V_L^2} \approx 16.4\ \text{V}$$

Question 5

The electric field of a plane traveling electromagnetic wave is given by

$$E_x = E_{\max}\cos(kz+\omega t),\ E_y = E_z = 0\,.$$

(a) State the direction of propagation. [1]

(b) Determine the magnitude and direction of $\vec{B}$. Express the maximum value of $\vec{B}$ in terms of $E_{\max}$. [3]

(c) Show that the average intensity – that is, the average rate of energy transport per unit area – of the above electromagnetic wave is given by

$$S_{\text{av}} = \frac{cB_{\max}^2}{2\mu_0}\,.\ [4]$$

Solution:

The direction of propagation is along the negative *z*-direction.

Since

$$\vec{E} = E_{\max}\cos(kz+\omega t)\hat{i}\,,$$

we have

$$\vec{B} = -B_{\max}\cos(kz+\omega t)\hat{j}$$

where

$$B_{\max} = \frac{1}{c}E_{\max}$$

The Poynting vector,

$$\vec{S} \equiv \frac{1}{\mu_0}\vec{E}\times\vec{B} = -\frac{1}{\mu_0}E_{\max}B_{\max}\cos^2(kz+\omega t)\hat{k}$$

It gives the energy flow per unit time per unit area

$$S = \frac{c}{\mu_0}B_{\max}^2\cos^2(kz+\omega t) = \frac{c}{2\mu_0}B_{\max}^2\left[1+\cos 2(kz+\omega t)\right]$$

The average rate of energy transport per unit area

$$S_{\text{av}} = \frac{cB_{\max}^2}{2\mu_0}$$

Question 6(a) (24 minutes)

A very long cylinder of radius R_1 and uniform charge density ρ has a cylindrical hole cut along its entire length so that the axes of the cylinder and the hole are parallel and separated by a distance *d* (Figure 4).

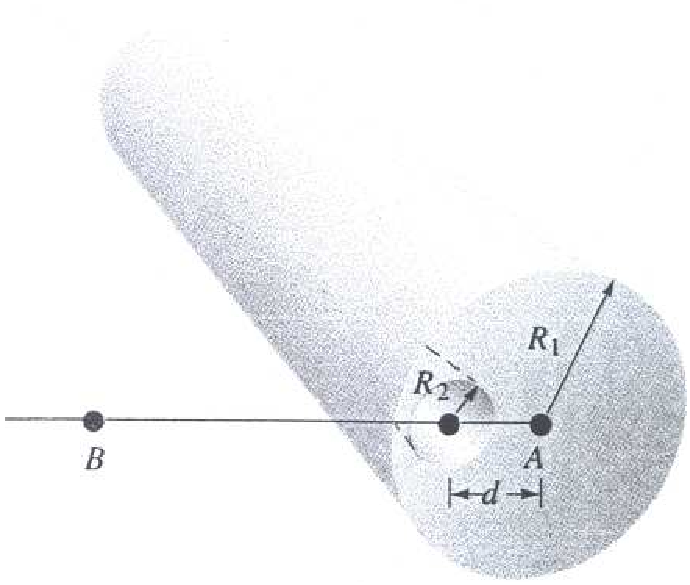


Figure 4

The radius of the hole is $R_2 < R_1$. Compute the electric field strength along the line *AB* in the following regions:

(i) $r \le d - R_2$.

(ii) $d - R_2 \le r \le d + R_2$.

(iii) $d + R_2 \le r \le R_1$. [8]

Solution:

According to Gauss's law,

$$\oint \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0} q_{\text{encl}}$$

For $r \le R_1$,

$$E_1 \times 2\pi r l = \frac{1}{\varepsilon_0} \times \pi r^2 l \times \rho \Rightarrow E_1(r) = \frac{\rho}{2\varepsilon_0} r \Rightarrow \vec{E}_1 = \frac{\rho}{2\varepsilon_0} \vec{r}$$

For $\left|\vec{r} - \vec{d}\right| \le R_2$,

$$E_2 \times 2\pi \left|\vec{r} - \vec{d}\right| l = \frac{1}{\varepsilon_0} \times \pi \left|\vec{r} - \vec{d}\right|^2 l \times (-\rho) \Rightarrow E_2\left(\left|\vec{r} - \vec{d}\right|\right) = -\frac{\rho}{2\varepsilon_0} \left|\vec{r} - \vec{d}\right| \Rightarrow \vec{E}_2 = -\frac{\rho}{2\varepsilon_0}\left(\vec{r} - \vec{d}\right)$$

For $\left|\vec{r}-\vec{d}\right| > R_2$,

$$E_2 \times 2\pi\left|\vec{r}-\vec{d}\right| l = \frac{1}{\varepsilon_0} \times \pi R_2^2 l \times (-\rho) \Rightarrow E_2\left(\left|\vec{r}-\vec{d}\right|\right) = -\frac{R_2^2 \rho}{2\varepsilon_0\left|\vec{r}-\vec{d}\right|} \Rightarrow \vec{E}_2 = -\frac{R_2^2 \rho}{2\varepsilon_0\left|\vec{r}-\vec{d}\right|^2}\left(\vec{r}-\vec{d}\right)$$

By the principle of superposition of electric fields, for $r \leq d - R_2$,

$$\vec{E}_1 + \vec{E}_2 = \frac{\rho}{2\varepsilon_0}\vec{r} - \frac{R_2^2 \rho}{2\varepsilon_0\left|\vec{r}-\vec{d}\right|^2}\left(\vec{r}-\vec{d}\right)$$

In particular, we have

$$\vec{E}_1 + \vec{E}_2 = -\frac{\rho}{2\varepsilon_0} r\hat{i} - \frac{R_2^2 \rho}{2\varepsilon_0(d-r)}\hat{i} = -\frac{\rho}{2\varepsilon_0}\left(r + \frac{R_2^2}{d-r}\right)\hat{i}$$

For $d - R_2 \leq r \leq d + R_2$, the electric field inside the cavity satisfies

$$\vec{E}_1 + \vec{E}_2 = \frac{\rho}{2\varepsilon_0}\vec{r} - \frac{\rho}{2\varepsilon_0}\left(\vec{r}-\vec{d}\right) = \frac{\rho}{2\varepsilon_0}\vec{d}$$

For $d + R_2 \leq r \leq R_1$,

$$\vec{E}_1 + \vec{E}_2 = -\frac{\rho}{2\varepsilon_0} r\hat{i} + \frac{R_2^2 \rho}{2\varepsilon_0(r-d)}\hat{i} = -\frac{\rho}{2\varepsilon_0}\left(r - \frac{R_2^2}{r-d}\right)\hat{i}$$

Question 6(b)

A spherical capacitor consists of two concentric, metallic spheres, as shown in Figure 5

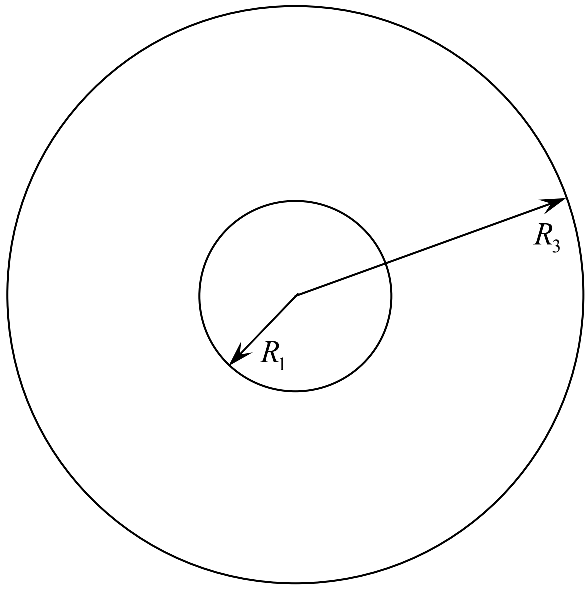


Figure 5

The inner sphere, of radius R_1, has charge $+Q$. The charge on the outer shell of radius R_3 $(R_1 < R_3)$ is $-Q$.

(i) Suppose the spheres are separated by a vacuum, find the capacitance. [6]

Solution:

Gauss's law

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Electric potential difference

$$V_{13} = \int_1^3 \vec{E} \cdot d\vec{l} = \frac{Q}{4\pi\varepsilon_0 R_1} - \frac{Q}{4\pi\varepsilon_0 R_3} = \frac{Q}{4\pi\varepsilon_0} \frac{R_3 - R_1}{R_1 R_3}$$

Capacitance

$$C_{13} = \frac{Q}{V_{13}} = 4\pi\varepsilon_0 \frac{R_1 R_3}{R_3 - R_1}$$

(ii) Now, suppose between the spheres are two concentric, spherical dielectrics of constants K_1 and K_2. As shown in Figure 6, the radius of the boundary between the two dielectrics is R_2 $(R_1 < R_2 < R_3)$.

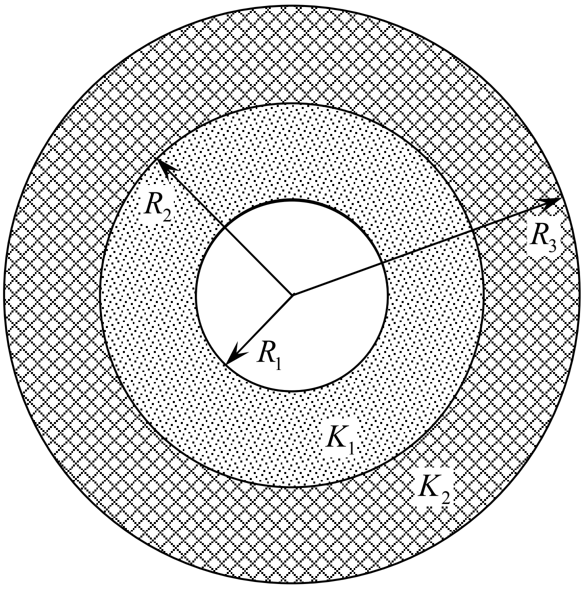


Figure 6

Compute the capacitance of the arrangement. [6]

Solution:

The combination is series

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{4\pi K_1 \varepsilon_0 \dfrac{R_1 R_2}{R_2 - R_1} \times 4\pi K_2 \varepsilon_0 \dfrac{R_2 R_3}{R_3 - R_2}}{4\pi K_1 \varepsilon_0 \dfrac{R_1 R_2}{R_2 - R_1} + 4\pi K_2 \varepsilon_0 \dfrac{R_2 R_3}{R_3 - R_2}}$$

$$= 4\pi\varepsilon_0 \frac{K_1 \dfrac{R_1 R_2}{R_2 - R_1} \times K_2 \dfrac{R_2 R_3}{R_3 - R_2}}{K_1 \dfrac{R_1 R_2}{R_2 - R_1} + K_2 \dfrac{R_2 R_3}{R_3 - R_2}}$$

$$= 4\pi\varepsilon_0 \frac{K_1 K_2 R_1 R_2^2 R_3}{K_1 R_1 R_2 (R_3 - R_2) + K_2 R_2 R_3 (R_2 - R_1)}$$

Question 7(a)

Consider a thin, straight wire of length $2a$, carrying a constant current I, and placed along the x axis as shown in Figure 7.

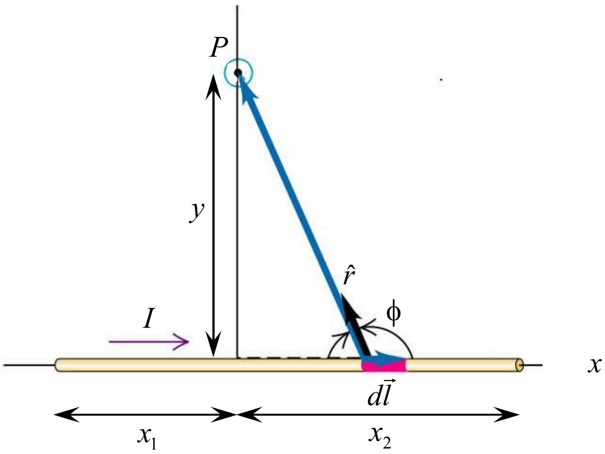


Figure 7

Determine the magnitude of the magnetic field at point P due to this current. Express your answer in terms of I, x_1, x_2, and y. [6]

Solution:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I}{r^2} d\vec{l} \times \hat{r}$$

$$\Rightarrow dB = \frac{\mu_0}{4\pi}\frac{I}{r^2} dx \sin\phi = \frac{\mu_0}{4\pi}\frac{I}{r^2} dx \sin(\pi - \phi) = \frac{\mu_0}{4\pi}\frac{I}{x^2+y^2}\frac{y}{\sqrt{x^2+y^2}} dx$$

$$B = \frac{\mu_0 I y}{4\pi}\int_{-x_1}^{x_2} \frac{dx}{\left(x^2+y^2\right)^{\frac{3}{2}}} = \frac{\mu_0 I}{4\pi y}\frac{x}{\sqrt{x^2+y^2}}\Bigg|_{-x_1}^{x_2} = \frac{\mu_0 I}{4\pi y}\left(\frac{x_1}{\sqrt{x_1^2+y^2}} + \frac{x_2}{\sqrt{x_2^2+y^2}}\right)$$

Question 7(b)

Compute the magnetic field strength at point P shown in Figure 8 in terms of the radius a , the length b, and the current I. [8]

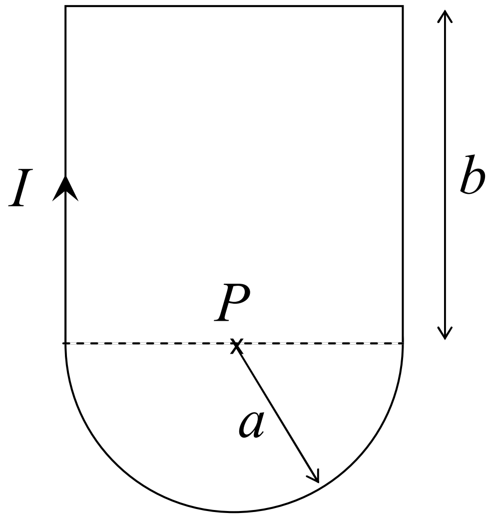


Figure 8

Solution:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I}{r^2}d\vec{l}\times\hat{r} \Rightarrow dB = \frac{\mu_0}{4\pi}\frac{I}{a^2}ad\theta \Rightarrow B_1 = \frac{\mu_0 I}{4a}$$

$$B_2 = \frac{\mu_0 I}{4\pi a}\left(\frac{b}{\sqrt{a^2+b^2}}\right) = B_4$$

$$B_3 = \frac{\mu_0 I}{4\pi b}\left(\frac{2a}{\sqrt{a^2+b^2}}\right)$$

$$B_1 + B_2 + B_3 + B_4 = \frac{\mu_0 I}{4a} + \frac{\mu_0 I}{2\pi ab}\sqrt{a^2+b^2}$$

Question 7(c)

A long, cylindrical conductor of radius b has a cylindrical cavity of radius a running parallel to its axis. The axis of the cavity is located a distance d from the axis of the cylinder as shown in Figure 9.

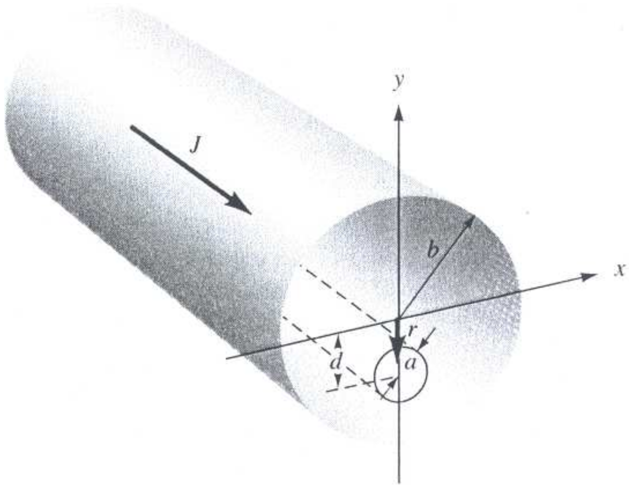


Figure 9

If the conductor carries a uniform current density J, compute the magnitude of the magnetic field strength within the cavity. [6]

Solution:

According to Ampere's law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i_{\text{encl}}$$

$$B_b \times 2\pi r = \mu_0 \times \pi r^2 \times J \Rightarrow B_b(r) = \frac{1}{2}\mu_0 J r$$

$$B_a \times 2\pi\left|\vec{r} - \vec{d}\right| = \mu_0 \times \pi\left|\vec{r} - \vec{d}\right|^2 \times (-J) \Rightarrow B_a\left(\left|\vec{r} - \vec{d}\right|\right) = -\frac{1}{2}\mu_0 J\left|\vec{r} - \vec{d}\right|$$

By the principle of superposition of magnetic fields, the magnetic field inside the cavity satisfies

$$B_\theta = B(r) + B\left(\left|\vec{r} - \vec{d}\right|\right)\cos\alpha = \frac{1}{2}\mu_0 J r - \frac{1}{2}\mu_0 J \frac{\vec{r} \cdot \left(\vec{r} - \vec{d}\right)}{r} = \frac{1}{2}\mu_0 J \hat{r} \cdot \vec{d}$$

$$B_r = B\left(\left|\vec{r} - \vec{d}\right|\right)\sin\alpha = -\frac{1}{2}\mu_0 J \frac{\left|\vec{r} \times \left(\vec{r} - \vec{d}\right)\right|}{r} = \frac{1}{2}\mu_0 J\left|\hat{r} \times \vec{d}\right|$$

where α is the angle between $\vec{r}$ and $\vec{r} - \vec{d}$. Therefore, the magnitude of the magnetic field strength within the cavity is given by

$$B = \frac{1}{2}\mu_0 J d$$

Question 8(a)

A rectangular loop of wire with dimensions l and w is released at $t = 0$ from rest just above a region in which the magnetic field is B_0 as shown in Figure 10.

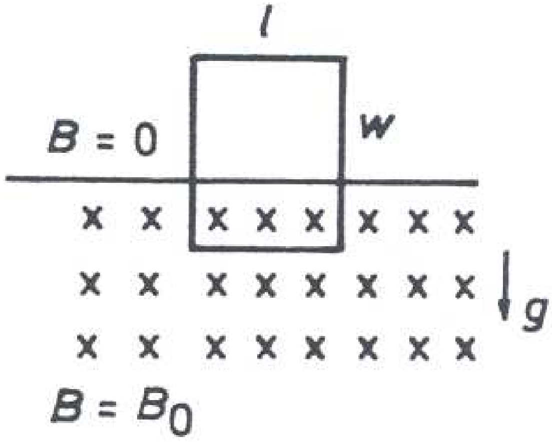


Figure 10

The loop has resistance R, self-inductance L, and mass m. Consider the loop during the time that it has its upper edge in the zero field region.

(i) Show that

$$mgv = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + \frac{d}{dt}\left(\frac{1}{2}Li^2\right) + i^2R\,,$$

where v is the instantaneous speed of the wire loop and i is the instantaneous current in it. [2]

(ii) Show that

$$m\frac{dv}{dt} = mg - B_0 il\,.\ \ [2]$$

(iii) Assume that the self-inductance can be ignored but not the resistance, find the current i and speed v of the loop as functions of time. [8]

Solution:

By conservation of energy, the rate of work done by the force of gravity equals the rate of change of the translational kinetic energy of the wire loop, the rate of change of the magnetic field energy, and the rate of energy dissipated in the wire loop:

$$mgv = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + \frac{d}{dt}\left(\frac{1}{2}Li^2\right) + i^2R$$

The force of gravtiy mg and the magnetic force due to interaction between the external magnetic field B_0 and current i are acting on the wire loop. Hence, v satisfies Newton's 2nd law

$$m\frac{dv}{dt} = mg - B_0 il \ .$$

The minus sign in front of $B_0 il$ is in accordance with Lenz's law.

If the self-inductance can be ignored,

$$mgv \approx \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + i^2R = mgv - iB_0lv + i^2R \Rightarrow iB_0lv \approx i^2R \Rightarrow i \approx \frac{B_0lv}{R}$$

And

$$m\frac{dv}{dt} = mg - \frac{B_0^2l^2}{R}v \Rightarrow \frac{dv}{mg - \frac{B_0^2l^2}{R}v} = \frac{1}{m}dt$$

$$-\frac{R}{B_0^2l^2}\ln\left(mg - \frac{B_0^2l^2}{R}v'\right)\Bigg|_0^v = \frac{1}{m}t$$

$$\ln\left(1 - \frac{B_0^2l^2}{mgR}v\right) = -\frac{B_0^2l^2}{mR}t \Rightarrow v = \frac{mgR}{B_0^2l^2}\left[1 - \exp\left(-\frac{B_0^2l^2}{mR}t\right)\right]$$

It follows that

$$i = \frac{B_0lv}{R} = \frac{mg}{B_0l}\left[1 - \exp\left(-\frac{B_0^2l^2}{mR}t\right)\right]$$

Question 8(b)

Figure 11 shows a circular parallel-plate capacitor being charged.

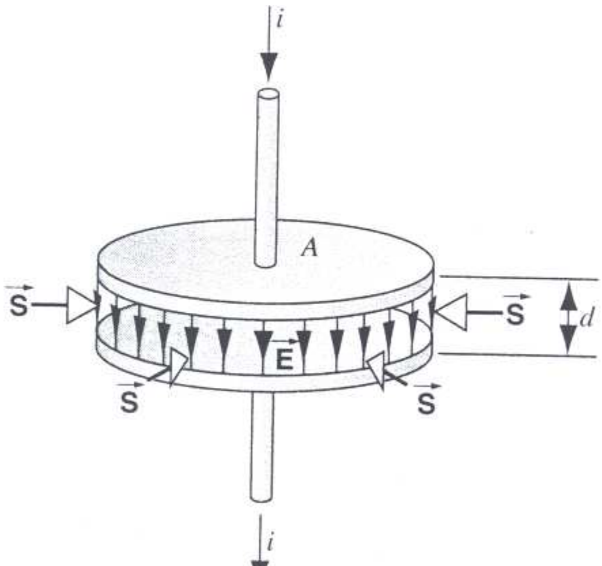


Figure 11

(i) Show that the Poynting vector $\vec{S}$ points eveywhere radially into the cylindrical volume. [2]

(ii) Show that the rate at which energy flows into this volume, calculated by integrating the Poynting vector over the cylindrical boundary of this volume, is equal to the rate at which the stored electrostatic energy increases; that is

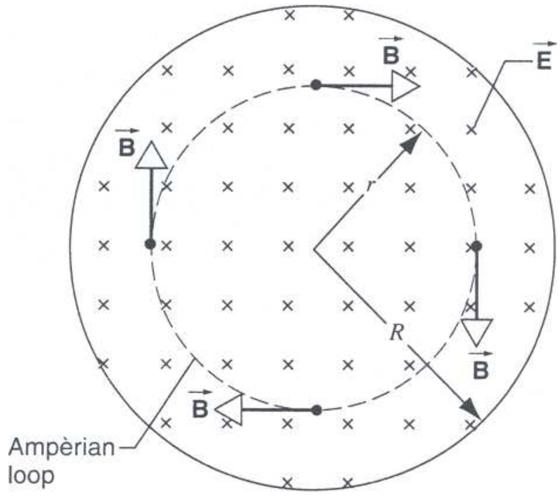
$$\int \vec{S}\cdot d\vec{A} = Ad\frac{d}{dt}\left(\frac{1}{2}\varepsilon_0 E^2\right). \quad [6]$$

Ignore fringing of $\vec{E}$.

Solution:

According to Ampere's law, $\oint \vec{B}\cdot d\vec{l} = \mu_0\varepsilon_0 \dfrac{d\Phi_E}{dt}$

Since the capacitor is being charged, $\dfrac{d\Phi_E}{dt}$ is positive. From the top view,



The Poynting vector, $\vec{S} \equiv \dfrac{1}{\mu_0}\vec{E}\times\vec{B}$ therefore points eveywhere radially into the cylindrical volume.

The Poynting vector gives the energy flow per unit time per unit area

$$S = \frac{1}{\mu_0}EB = \frac{\varepsilon_0 A}{2\pi r}E\frac{dE}{dt} = \frac{A}{2\pi r}\frac{d}{dt}\left(\frac{1}{2}\varepsilon_0 E^2\right)$$

since Ampere's law yields

$$B\times 2\pi r = \mu_0\varepsilon_0 A\frac{dE}{dt} \Rightarrow B = \frac{\mu_0\varepsilon_0 A}{2\pi r}\frac{dE}{dt}$$

Therefore,

$$\int \vec{S}\cdot d\vec{A} = \frac{A}{2\pi r}\frac{d}{dt}\left(\frac{1}{2}\varepsilon_0 E^2\right)\times 2\pi r d = Ad\frac{d}{dt}\left(\frac{1}{2}\varepsilon_0 E^2\right)$$