PC1143 2011/2012 Exam Solutions

Question 1

a) Assumption: shells are conductors.

Notes: the system given is a capacitor. Make use of spherical symmetry.

Energy density, $u = \frac{1}{2}\varepsilon_0 E^2$. in this case E means electric field strength.

To find E, we use gauss law.

$$\int E \cdot d\vec{A} = \frac{q_{encl}}{\epsilon_0}$$

$$E(4\pi r^2) = \frac{Q}{\varepsilon_0}$$

$$E = \frac{1}{4\pi r^2}\frac{Q}{\varepsilon_0}$$

$$u = \frac{1}{2}\varepsilon_0 \frac{1}{4\pi r^2}\frac{Q^2}{\varepsilon_0^2} = \frac{1}{2}\frac{Q^2}{\varepsilon_0 4^2\pi^2 r^4}$$

$$U_0 = \int u\, dV$$

where V represents volume.

$$U_0 = \int \frac{1}{2}\frac{Q^2}{4^2\pi^2\varepsilon_0 r^4}\, 4\pi r^2\, dr = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\int \frac{1}{r^2} dr = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_a^b = \frac{1}{8}\frac{Q^2}{\pi\varepsilon_0}\left(-\frac{1}{b}+\frac{1}{a}\right)$$

Remember to include upper and lower limit when doing integration!

b) $U = \frac{1}{8}\frac{Q^2}{\pi k\varepsilon_0}\left(-\frac{1}{b}+\frac{1}{a}\right)$. k represents dielectric constant.

c) U is smaller than U_0, since there are induced charges on the dielectric surface and work is done by system to induce these charges, leading to a decrease in stored energy.

Question 2

a) $F_B = \frac{mv^2}{r}$

$$Bqv = \frac{mv^2}{r}$$

$$B(1.60 \times 10^{-19}) = \frac{9.11 \times 10^{-31} \times 0.01c}{10^{-8} \times 10^{-2}}$$

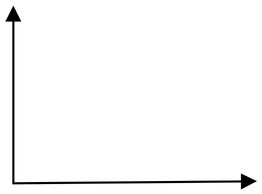
$$B = 1.71 \times 10^5 \mathrm{T}$$

b) i) Use $B = \frac{\mu_0}{2\pi r}$. Using the right hand grip rule, the magnetic fields due to the wires carrying current I cancel each other out. Thus, we only need to consider magnetic field produced by wire carrying current $2I$.

$$r = \sqrt{\frac{d^2}{4}+\frac{d^2}{4}} = \frac{d}{\sqrt{2}}$$

$$B = \frac{\mu_0 2I}{2\pi r} = \frac{\mu_0 2I}{2\pi \frac{d}{\sqrt{2}}} = \frac{\mu_0 I\sqrt{2}}{\pi d}$$

b) ii)



Arrows defining positive x and positive y.

$$B_{2I} = \frac{\mu_0 2I}{2\pi r}$$

$$r = \sqrt{d^2 + d^2} = \sqrt{2}d$$

$$B_{2I} = \frac{\mu_0 2I}{2\pi\sqrt{2}d}$$

$$B_{2Ix} = -\frac{\mu_0 2I}{2\pi\sqrt{2}d}\sin 45° = -\frac{\mu_0 I}{\pi\sqrt{2}d\sqrt{2}} = -\frac{\mu_0 I}{2\pi d}$$

$$B_{2Iy} = -\frac{\mu_0 2I}{2\pi\sqrt{2}d}\cos 45° = -\frac{\mu_0 I}{2\pi d}$$

$$B_{Ix} = \frac{\mu_0 I}{2\pi d}$$

$$B_{Iy} = \frac{\mu_0 I}{2\pi d}$$

$$B_x = \frac{\mu_0 I}{2\pi d} - \frac{\mu_0 I}{2\pi d} = 0$$

$$B_y = \frac{\mu_0 I}{2\pi d} - \frac{\mu_0 I}{2\pi d} = 0$$

So, $B = 0$.

Question 3

a) At steady state, before the switch is flipped:

By loop rule, $-iR_2 - L\frac{di}{dt} + \mathcal{E}_0 = 0$

At steady state, $\frac{di}{dt} = 0$

$$\mathcal{E}_0 = iR_2$$

$$i(t=0) = \frac{\mathcal{E}_0}{R_2}$$

After the switch is flipped from a to b:

By loop rule, $-R_1 i - L\frac{di}{dt} = 0$

$$\frac{di}{dt} = -\frac{R_1}{Li}$$

$$i(t) = i(t=0)e^{-\frac{R_1 t}{L}} = \frac{\mathcal{E}_0}{R_2}e^{-\frac{R_1 t}{L}}$$

Must memorise solution to above differential equation!

b) Power dissipated in resistor $= i^2 R_1 = \frac{\mathcal{E}_0^2}{R_2^2} e^{-\frac{2R_1}{L}t} R_1$

Total energy dissipated $= \int P\, dt = \frac{\mathcal{E}_0^2}{R_2^2} R_1 \int e^{-\frac{2R_1}{L}t}\, dt = \frac{\mathcal{E}_0^2}{R_2} R_1 e^{-\frac{2R_1}{L}t} \left(-\frac{L}{2R_2}\right)$

Integrate from $t = 0$ to $t = \infty$, total energy dissipated,

$\frac{\mathcal{E}_0^2}{R_2^2} \frac{L}{2R_1} R_1 = \frac{1}{2} L \frac{\mathcal{E}_0^2}{R_2^2} = \frac{1}{2} L[i(t=0)]^2 =$ Energy stored in inductor at $t = 0$. (shown)

Question 4

a) For RLC is parallel, i_R is the same phase as the source voltage. $v_R = v_L = v_C = v_{source}$.

$$i_R = I_R \sin(\omega t) = \frac{V}{R} \sin(\omega t)$$

i_L lag i_R by $\frac{\pi}{2}$ radian (Note: must know this for exam!)

$$i_L = -\frac{V}{L\omega} \cos(\omega t)$$

i_C lead i_R by $\frac{\pi}{2}$ radian (Note: must know this for exam!)

$$i_C = I_C cos\ (\omega t) = \frac{V}{X_C} \cos(\omega t) = CV\omega \cos(\omega t)$$

b) Power dissipated in circuit = Power dissipated in resistor $= i_R^2 R = \frac{V^2}{R^2} \sin^2(\omega t)\, R = \frac{V^2}{R} \sin^2(\omega t)$

c) Energy stored in circuit = Energy in capacitor + Energy in inductor

$$= \frac{1}{2} CV^2 + \frac{1}{2} L i_L^2$$
$$= \frac{1}{2} CV^2 \sin^2(\omega t) + \frac{1}{2} L \frac{V^2}{L^2 \omega^2} \cos^2(\omega t)$$
$$= \frac{1}{2} CV^2 \sin^2(\omega t) + \frac{1}{2} \frac{V^2}{L\omega^2} \cos^2(\omega t)$$

Question 5

a) Concept: Direction of propagation is same as direction of $E \times B$.

Direction of propagation is $-\hat{\imath}$ direction (given) and E is in $\hat{\jmath}$ direction, since it is given that E is perpendicular to $\hat{k}$ direction and it must also be perpendicular to the $-\hat{\imath}$ direction.

Note: $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ in solution to this question refer to unit vectors.

$-\hat{\imath} = \hat{\jmath} \times -\hat{k}$

$\vec{E}(x,t) = E_{max} \cos(kx + \omega t)\, \hat{\jmath} = cB_{max} \cos(kx + \omega t)\, \hat{\jmath}$

$\vec{B}(x,t) = -B_{max} \cos(kx + \omega t)\, \hat{k}$

Find ω using $\omega = 2\pi f$, f is given in the question to be 100MHz. Find k using $k = \frac{\omega}{c}$.

b) $S = \frac{1}{\mu_0} E \times B$

$S = 1/\mu_0 E_{max}\cos(kx+\omega t)\,\hat{j}\times(-B_{max}\cos(kx+\omega t))\hat{k} = -\frac{1}{\mu_0}E_{max}B_{max}\cos^2(kx+\omega t)\,\hat{\imath}$

c) $I = S_{av} = \frac{1}{2\mu_0}E_{max}B_{max} = \frac{1}{2\mu_0}cB_{max}B_{max}$
Given $I = 1000\text{W/m}^2$, and given μ_0 and c, we can solve for B_{max}.
Then, find E_{max} with formula $E_{max} = cB_{max}$.

Question 6

a) i) In this solution, $\hat{\imath}$ and $\hat{k}$ refer to unit vectors.

$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

$r = x\hat{\imath} - z\hat{k}$

$r^2 = x^2 + z^2$

Unit vector, $r = \frac{x\hat{\imath}-z\hat{k}}{\sqrt{x^2+z^2}}$

$dE = \frac{1}{4\,\pi\varepsilon_0}\frac{dq}{x^2+z^2}\frac{x\hat{\imath}-z\hat{k}}{\sqrt{x^2+z^2}}$

$dq = \frac{dz}{2l}Q$

$dE = \frac{1}{4\,\pi\varepsilon_0}\frac{Q}{2l}\frac{x\hat{\imath}-z\hat{k}}{(\,x^2+z^2)^{\frac{3}{2}}}\,dz$

$E_x = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2l}\int_{-l}^{l}\frac{x}{(\,x^2+z^2)^{\frac{3}{2}}}\,dz$

$E_x = \frac{Qx}{8\pi\varepsilon_0 l}\left(\frac{1}{x^2}\frac{z}{\sqrt{x^2+z^2}}\right)$

$E_x = \frac{Qx}{8\pi\varepsilon_0 l}\frac{1}{x^2}\frac{2l}{\sqrt{x^2+l^2}} = \frac{2Q}{8\pi\varepsilon_0 x\sqrt{x^2+l^2}}$

Because charge distribution is symmetrical about x axis, net $E_z = 0$.

a) ii) When line of charge is infinitely long, $l \gg x$,

$E_x = \frac{Q}{4\pi\varepsilon_0 x\sqrt{x^2+l^2}} = \frac{Q}{4\pi\varepsilon_0 x\sqrt{l^2}\left(1+\sqrt{\left(\frac{x}{l}\right)^2}\right)} \approx \frac{Q}{4\pi\varepsilon_0 xl} = \frac{\lambda}{4\pi\varepsilon_0 x}$

where $\lambda = \frac{Q}{l}$.

b) i) $C = \frac{Q}{V}$. Consider Gauss' Law, $\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$,

$E(2\pi rL) = \frac{Q}{\varepsilon_0}$, where Q is the charge on one cylinder.

$E = \frac{1}{2\pi rL}\frac{Q}{\varepsilon_0}$

Since $E = -\frac{dV}{dr}$, $dV = -E\,dr$,

$$dV = -\frac{Q}{2\pi r L\varepsilon_0} dr$$
$$\int dV = -\frac{Q}{2\pi L\varepsilon_0} \int \frac{1}{r} dr$$
Take integration limits from $r = a$ to $r = b$,
$$V_a - V_b = \frac{Q}{2\pi L\varepsilon_0} \ln\left(\frac{b}{a}\right)$$
$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{2\pi L\varepsilon_0}} \ln\left(\frac{b}{a}\right) = 2\pi L\varepsilon_0 \ln\left(\frac{b}{a}\right)$$

b) ii) Given $b - a \ll a$. Consider the very thin cylindrical strip, by Gauss' Law, $\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$,
$$E(2\pi a L) = \frac{Q}{\varepsilon_0}$$
$$\frac{dV}{dr}(2\pi a L) = \frac{Q}{\varepsilon_0}$$
$$dV = \frac{Q}{2\pi\varepsilon_0 a L} dr$$
$$V = \frac{Q}{2\varepsilon_0 \pi a L}(b - a)$$
$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{2\varepsilon_0 \pi a L}}(b - a) = 2\varepsilon_0 \pi a L(b - a)$$
But I don't know how to comment on my answer. =(

Question 7

a) i) Making use of Biot-Savart's Law:
$$dB = \frac{\mu_0}{4\pi} \frac{1}{r^2} dl \times \hat{r}$$
Let $\hat{r}_2 = a\cos\theta\, \hat{\imath} + a\sin\theta\, \hat{\jmath}$
$$dl = -a\sin\theta\, d\theta \hat{\imath} + a\cos\theta\, d\theta \hat{\jmath}$$
$$\hat{r}_1 = z\hat{k}$$
$$\vec{r} = \vec{r}_1 - \vec{r}_2 = -a\cos\theta\, \hat{\imath} - a\sin\theta\, \hat{\jmath} + z\hat{k}$$
$$r^2 = a^2 + z^2$$
Unit vector or $\vec{r}$, $r = \frac{1}{\sqrt{a^2+z^2}}\left(-a\cos\theta\, \hat{\imath} - a\sin\theta\, \hat{\jmath} + z\hat{k}\right)$
$$dB = \frac{\mu_0}{4\pi} \frac{I}{a^2 + z^2} \left(-a\sin\theta\, d\theta \hat{\imath} + a\cos\theta\, d\theta \hat{\jmath}\right) \times \frac{-a\cos\theta\, \hat{\imath} - a\sin\theta\, \hat{\jmath} + z\hat{k}}{\sqrt{a^2 + z^2}}$$
$$= \frac{\mu_0}{4\pi} \frac{I}{(a^2 + z^2)^{\frac{3}{2}}} \left(a^2\sin^2\theta\, d\theta \hat{k} + az\sin\theta\, d\theta \hat{\jmath} + a^2\cos^2\theta\, d\theta \hat{k} + az\cos\theta\, d\theta \hat{\imath}\right)$$
$$= \frac{\mu_0}{4\pi} \frac{I}{(a^2 + z^2)^{\frac{3}{2}}} \left(a^2 d\theta \hat{k} + az\sin\theta\, d\theta \hat{\jmath} + az\cos\theta\, d\theta \hat{\imath}\right)$$
$$B = \int_0^{2\pi} dB = \frac{\mu_0}{4\pi} \frac{I}{(a^2 + z^2)^{\frac{3}{2}}} \int_0^{2\pi} a^2\, d\theta \hat{k} = \frac{\mu_0}{4\pi} \frac{I}{(a^2 + z^2)^{\frac{3}{2}}} 2\pi \hat{k} = \frac{1}{2} \frac{\mu_0 I}{(a^2 + z^2)^{\frac{3}{2}}} \hat{k}$$

a) ii) $\int_{-L}^{L} B_z \, dz = \frac{\mu_0 I a^2}{2} \int_{-L}^{L} \frac{1}{(a^2+z^2)^{\frac{3}{2}}} dz = \frac{\mu_0 I a^2}{2} \frac{1}{a^2} \left[\frac{z}{\sqrt{a^2+z^2}}\right]_{-L}^{L}$

$$\int_{-L}^{L} B_z \, dz = \mu_0 \frac{1}{2} \frac{2L}{\sqrt{a^2+L^2}} = \frac{\mu_0 I L}{\sqrt{a^2+L^2}}$$

For L approaching ∞:

$$\frac{\mu_0 I L}{\sqrt{a^2+L^2}} = \frac{\mu_0 I L}{\sqrt{L^2}} \frac{1}{\sqrt{1+\left(\frac{a}{L}\right)^2}} \approx \mu_0 I$$

b) $B = \frac{\mu_0 I n}{2} a^2 \int_{z-\frac{l}{2}}^{z+\frac{l}{2}} \frac{1}{(a^2+m^2)^{\frac{3}{2}}} dm = \frac{\mu_0 I n}{2} a^2 \frac{1}{a^2} \left[\frac{m}{\sqrt{a^2+m^2}}\right]_{z-\frac{l}{2}}^{z+\frac{l}{2}} = \frac{\mu_0 I n}{2} \left[\frac{z+\frac{l}{2}}{\sqrt{a^2+\left(z+\frac{l}{2}\right)^2}} - \frac{z-\frac{l}{2}}{\sqrt{a^2+\left(z-\frac{l}{2}\right)^2}}\right] \hat{k}$

c) i) $\hat{\imath}$ and $\hat{\jmath}$ represent unit vectors. At point Q,

$\vec{r}_2 = y\hat{\jmath}$

$\vec{r}_1 = a\cos\theta\,\hat{\imath} + a\sin\theta\,\hat{\jmath}$

$d\vec{l} = -a\sin\theta\, d\theta\hat{\imath} + a\cos\theta\, d\theta\hat{\jmath}$

Using Biot-Savart's Law,

$$dB = \frac{\mu_0}{4\pi} \frac{1}{r^2} dl \times \hat{r}$$

$\vec{r} = \vec{r}_2 - \vec{r}_1 = -a\cos\theta\,\hat{\imath} + (y - a\sin\theta)\hat{\jmath}$

For point Q far away from loop,

$\vec{r} = -a\cos\theta\,\hat{\imath} + y\hat{\jmath}$

$r = \sqrt{a^2\cos^2\theta + y^2} = y$, since point on y axis is very far away from loop.

$$\hat{r} = \frac{1}{y}(-a\cos\theta\,\hat{\imath} + y\hat{\jmath})$$

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I}{y^2}(-a\sin\theta\, d\theta\hat{\imath} + a\cos\theta\, d\theta\hat{\jmath}) \times \frac{1}{y}(-a\cos\theta\,\hat{\imath} + y\hat{\jmath})$$

$$= \frac{\mu_0}{4\pi} \frac{I}{y^3}\left(-ay\sin\theta\, d\theta\hat{k} + a^2\cos^2\theta\, d\theta\hat{k}\right)$$

$$B = \frac{\mu_0}{4\pi} \frac{I}{y^3} \int_0^{2\pi} -ay\sin\theta + a^2\cos^2\theta\, d\theta$$

$$= \frac{\mu_0}{4\pi} \frac{Ia^2}{y^3} \int_0^{2\pi} \cos^2\theta\, d\theta$$

$$= \frac{\mu_0}{4\pi} \frac{Ia^2}{y^3} \int_0^{2\pi} \frac{\cos 2\theta + 1}{2} d\theta$$

$$= \frac{\mu_0}{4} \frac{Ia^2}{y^3}$$

For point very far away from loop, y is very large. $B = 0$.

c) ii) Ampere's Law:

$$\int \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

Consider a closed loop from $z = -\infty$ to ∞, from $y = 0$ to point Q.

Consider $\vec{B} \cdot d\vec{l}$ over this closed loop,

$$\int_{-\infty}^{\infty} B_z \, dz = \mu_0 I$$

This is because the system is symmetric about y axis, so $\vec{B} \cdot d\vec{l}$ on line above y axis cancels out with $\vec{B} \cdot d\vec{l}$ on line below y axis, and $\vec{B} \cdot d\vec{l}$ on line cutting through Q is 0 because we approximate $B = 0$ at any point along this line.

Question 8

a) i) By Faraday's Law,

$$E = -\frac{d\Phi}{dt}$$

In this case, $\Phi = \frac{B_0}{\sqrt{2}}\left(1 - e^{-\lambda t}\right)\pi a^2$

Note: consider B in $\hat{k}$ direction only, because B in $\hat{j}$ direction doesn't contribute to flux.

$$E = -\frac{d}{dt}\left[\frac{B_0}{\sqrt{2}}\left(1 - e^{-\lambda t}\right)\pi a^2\right] = -\frac{\pi a^2 B_0}{\sqrt{2}}\left[-e^{-\lambda t}(-\lambda)\right] = -\frac{\pi a^2 B_0 \lambda e^{-\lambda t}}{\sqrt{2}}$$

$$R_{eff} = R_1 + R_2$$

$$I(t) = \frac{E}{R_{eff}} = \frac{\pi a^2 B_0}{\sqrt{2}} \frac{\lambda e^{-\lambda t}}{R_1 + R^2}$$

a) ii) Draw a graph, same shape as $y = e^{-x}$ graph. Must say that asymptote is at $I(t) = 0$ and $I(t)$ intercept is $\frac{\pi a^2 B_0}{\sqrt{2}} \frac{\lambda}{R_1 + R_2}$.

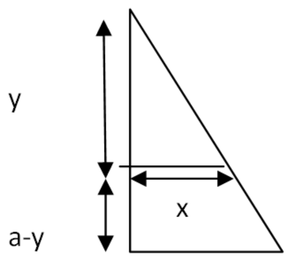
a) iii) $V_1 = IR_1 = \frac{\pi a^2 B_0}{\sqrt{2}} e^{-\lambda t} \frac{\lambda R_1}{R_1 + R_2}$

$$V_2 = IR_2 = \frac{\pi a^2 B_0}{\sqrt{2}} e^{-\lambda t} \frac{\lambda R_2}{R_1 + R_2}$$

a) iv) The emf induced as a result of changing magnetic flux through wire loop is non conservative and depends on path taken. Even if $R_1 = R_2$, with respect to R_1, current flows from q to p, q has higher potential. With respect to R_2, current flows from p to q, p has higher potential. So the readings on both voltmeters will be different (have different sign).

b) Use the definition: flux through triangle loop = mutual inductance × current through straight wire

Consider similar triangles:



From trigonometry ratios, $\tan 60° = \sqrt{3} = \frac{y}{x}$

$x = \frac{y}{\sqrt{3}}$

Consider many horizontal rectangular strips,

$dA = 2x\,dy = 2\frac{y}{\sqrt{3}}\,dy$

B cutting through the rectangular strip, $B = \frac{\mu_0 I}{2\pi[b+(a-y)]}$.

Flux through the triangle,

$$\int \vec{B}\cdot d\vec{A} = \int \frac{\mu_0 I}{2\pi[b+(a-y)]}\;2\frac{y}{\sqrt{3}}dy$$
$$= \frac{\mu_0 I}{\pi\sqrt{3}}\int_0^a \frac{y}{b+a-y}dy$$
$$= \frac{\mu_0 I}{\pi\sqrt{3}}\int_0^a \frac{y-b-a+b+a}{b+a-y}dy$$
$$= \frac{\mu_0 I}{\pi\sqrt{3}}\int_0^a \frac{b+a}{b+a-y} - 1\,dy$$
$$= \frac{\mu_0 I}{\pi\sqrt{3}}[-(b+a)\ln(b+a-y) - y]_0^a$$
$$= \frac{\mu_0 I}{\pi\sqrt{3}}\left[-a + (b+a)\ln\left(\frac{b+a}{b}\right)\right]$$

Using the definition: flux through triangle loop = mutual inductance × current through straight wire

$$\frac{\mu_0 I}{\pi\sqrt{3}}\left[-a + (b+a)\ln\left(\frac{b+a}{b}\right)\right] = MI$$
$$M = \frac{\mu_0}{\pi\sqrt{3}}\left[-a + (b+a)\ln\left(\frac{b+a}{b}\right)\right]$$

Answers provided by : **Woo Jia QIan**
Formatted and edited by : **John Soo**