

NUS PHYSICS SOCIETY

PAST YEAR PAPER SOLUTIONS

PC 1143 : AY 2012 /13

Compiled by:

NUS Physics Society Past Year Solution Team

- Yeo Zhen Yuan
- Ng Kian Fong
- Leong Jin Feng
- Ryan Goh

Special Thanks to:

- Alex Hue Jun Hao
- Wesley Wong Lee Wei
- Tang Zong Sheng

These solutions were verified by NUS Physics Alumni.

Published on: April 21, 2015

Problem One

A spherical capacitor C has an inner sphere of radius R_1 , with a charge of $+Q$ and an outer concentric spherical shell of radius R_2 with a charge of $-Q$.

- (a) Find the energy density associated with the electric field at any point in space between the two conductors. [4m]

Solution:

Owing to the spherical symmetry of the system, we consider spherical Gaussian surfaces of radii r that are concentric with the spherical capacitor. Utilising Gauss's Law, and making use of the fact that the electric field \mathbf{E} is radially directed, we then have

$$\oiint \mathbf{E} \cdot d\mathbf{A} = E (4\pi r^2) = \frac{Q_{\text{enc}}}{\epsilon_0}$$

Hence, the electric field is given by

$$\mathbf{E}(r) = \begin{cases} 0 & \text{for } r \leq R_1 \\ \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}} & \text{for } R_1 < r < R_2 \\ 0 & \text{for } r \geq R_2 \end{cases}$$

The energy density stored inside the electric field in the region between the two conductors is then given by

$$u_E = \frac{1}{2} \epsilon_0 (\mathbf{E} \cdot \mathbf{E}) = \frac{1}{2} \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 r^2} \right)^2 = \frac{Q^2}{32\pi^2 \epsilon_0 r^4}$$

- (b) Hence, or otherwise, find the capacitance of C . [4m]

Solution:

The total energy stored in the spherical capacitor U can be obtained by integrating the energy density derived in part (a) over the region between the two spherical surfaces:

$$\begin{aligned} U &= \int_{R_1}^{R_2} \frac{Q^2}{32\pi^2 \epsilon_0 r^4} (4\pi r^2) dr = \int_{R_1}^{R_2} \frac{Q^2}{8\pi\epsilon_0 r^2} dr \\ &= -\frac{Q^2}{8\pi\epsilon_0 r} \Big|_{R_1}^{R_2} = \frac{Q^2}{8\pi\epsilon_0} \frac{R_2 - R_1}{R_1 R_2} \end{aligned}$$

Recall also that the energy of a capacitor is related to its capacitance C by

$$U = \frac{Q^2}{2C}.$$

By equating the two expressions for U and making C the subject, we finally obtain:

$$\frac{Q^2}{2C} = \frac{Q^2}{8\pi\epsilon_0} \frac{R_2 - R_1}{R_1 R_2} \quad \Rightarrow \quad C = 4\pi\epsilon_0 \frac{R_1 R_2}{R_2 - R_1}$$

Problem Two

Of the total energy drawn from a battery of emf \mathcal{E} in charging a capacitor of capacitance C via a resistor of resistance R , show that half ends up dissipated in the resistor. [8m]

Solution:

By applying Kirchoff's loop rule, we find that

$$\mathcal{E} - \frac{q}{C} - R \frac{dq}{dt} = 0,$$

recognising that the current is given by the rate of change of charge with time. The differential equation for q can be solved via separation of variables:

$$\int_0^{q(t)} \frac{dq}{C\mathcal{E} - q} = \int_0^t \frac{1}{RC} dt \quad \Rightarrow \quad \ln \left| \frac{C\mathcal{E} - q(t)}{C\mathcal{E}} \right| = -\frac{t}{RC}$$

This yields

$$q(t) = C\mathcal{E} \left(1 - \exp \left[\frac{-t}{RC} \right] \right) \quad \text{and} \quad I(t) = \frac{dq}{dt} = \frac{\mathcal{E}}{R} \exp \left[\frac{-t}{RC} \right].$$

The total energy dissipated in the resistor is then given by

$$\begin{aligned} E_R &= \int_0^\infty P dt = \int_0^\infty I(t)^2 R dt = \int_0^\infty \frac{\mathcal{E}^2}{R} \exp \left[\frac{-2t}{RC} \right] dt \\ &= -\frac{C\mathcal{E}^2}{2} \exp \left[\frac{-2t}{RC} \right] \Big|_0^\infty \\ &= \frac{C\mathcal{E}^2}{2} \end{aligned}$$

Meanwhile, the total work done by the battery is

$$E = Q\mathcal{E} = C\mathcal{E}^2.$$

Hence, we have shown that

$$E_R = \frac{1}{2}E,$$

that is, half the work done by the battery is dissipated in the resistor.

Problem Three

A long straight wire and a small rectangular wire loop lie in the same plane, as illustrated in Figure 1 (refer to the exam paper). Determine the *mutual inductance* in terms of l_1 , l_2 and w . Assume the wire is very long and the rest of the circuit is very far away compared to l_1 , l_2 and w . [8m]

Solution:

Assume that a current I is flowing through the long straight wire towards the right. The magnetic field \mathbf{B} at a perpendicular distance r from the long straight wire is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi},$$

where $\hat{\phi}$ is pointing into the page.

The total magnetic flux Φ_B through the wire loop is then given by

$$\begin{aligned} \Phi_B &= \int \mathbf{B} \cdot d\mathbf{A} = \int_{l_1}^{l_2} \frac{\mu_0 I}{2\pi r} w \, dr \\ &= \frac{\mu_0 I w}{2\pi} \ln(l_2/l_1). \end{aligned}$$

Finally, the mutual inductance between the wire and the loop is

$$M = \Phi_B/I = \frac{\mu_0 w}{2\pi} \ln(l_2/l_1).$$

Problem Four

A series circuit consists of a 100Ω resistor, a 36mH inductor, and a 4.0nF capacitor. The circuit is connected to an ac source with constant voltage amplitude 20V , and whose frequency can be varied over a wide range.

- (a) Find the resonance frequency f_0 of the circuit. [2m]

Solution:

The resonance frequency of a series LRC circuit is given by

$$\omega_0 = 2\pi f_0 = \sqrt{\frac{1}{LC}}.$$

Evaluating numerically, we obtain

$$f_0 = 1.326 \times 10^4 \text{Hz}.$$

- (b) At resonance, what is the rms current in the circuit and what are the rms voltage across the inductor and capacitor? [6m]

Solution:

At resonance, the impedance of the circuit is simply given by $Z = R$ i.e. entirely due to the resistor alone. Hence, the rms current in the circuit is simply

$$I_{rms} = \frac{V_{rms}}{R} = \frac{20}{100\sqrt{2}} = 0.141\text{A}$$

The rms voltages across the inductor and capacitor are respectively given by

$$V_{rms,L} = \omega L I_{rms} = 422.9\text{V} \quad \text{and} \quad V_{rms,C} = (1/\omega C) I_{rms} = 422.9\text{V}.$$

Problem Five

The electric field of an electromagnetic wave oscillates in the z-direction and the Poynting vector is given by

$$\mathbf{S}(x, t) = (100\text{W/m}^2)\sin^2[10x - (3.0 \times 10^9)t]\hat{\mathbf{i}}.$$

- (a) What is the direction of propagation of the wave? [1m]

Solution:

The direction of propagation of the wave is obviously in the direction of the Poynting vector since it represents the directional energy flux density delivered by the electromagnetic wave. Hence the wave is propagating in the positive x -direction.

- (b) Find its wavelength and the frequency. [2m]

Solution:

By comparing the expression of the Poynting vector of the wave with

$$\mathbf{S}(x, t) = S_0 \sin^2[kx - \omega t]\hat{\mathbf{i}},$$

we find that $k = 10 \text{ m}^{-1}$ and $\omega = 3.0 \times 10^9 \text{ rad/s}$. Hence the wavelength and frequency of the wave are given by

$$\lambda = \frac{2\pi}{k} = 0.628 \text{ m} \quad \text{and} \quad f = \omega/2\pi = 477.5 \times 10^6 \text{ Hz}.$$

- (c) Find the electric and magnetic fields. [5m]

Solution:

Recall that the Poynting vector is given by

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B},$$

and hence

$$S_0 = \frac{1}{\mu_0} E_0 B_0 = \frac{1}{\mu_0} E_0 (E_0/c) = \frac{E_0^2}{c\mu_0} = 100 \text{ W/m}^2.$$

It is then straightforward to obtain $E_0 = 194.2 \text{ Vm}^{-1}$ and $B_0 = 647.2 \text{ nT}$. Finally, the electric and magnetic fields are

$$\begin{aligned} \mathbf{E}(x, t) &= (194.2 \text{ Vm}^{-1}) \sin(10x - (3.0 \times 10^9)t) \hat{\mathbf{k}} \\ \mathbf{B}(x, t) &= -(647.2 \text{ nT}) \sin(10x - (3.0 \times 10^9)t) \hat{\mathbf{j}}, \end{aligned}$$

where the relevant directions of the fields are obtained from the right-hand (cross-product) rule.

Problem Six

- (a) An electric dipole is centred at origin O , with electric dipole moment \mathbf{p} in the direction of the x-axis (Refer to Figure 2 on the past year question paper). Derive an approximate expression for the electric field at a point P on the x-axis for which x is much larger than d . Express your answer in terms of \mathbf{p} . [8m]

Solution:

The resultant electric field at point P is the superposition of the electric fields from each of the two charges:

$$\begin{aligned} E_P &= \frac{kq}{(x - \frac{d}{2})^2} - \frac{kq}{(x + \frac{d}{2})^2} = kq \left[\left(x - \frac{d}{2}\right)^{-2} - \left(x + \frac{d}{2}\right)^{-2} \right] \\ &= \frac{kq}{x^2} \left[\left(1 + \frac{d}{x} + \dots\right) - \left(1 - \frac{d}{x} + \dots\right) \right] && \text{Taylor's expansion} \\ &\approx \frac{kq}{x^2} \left[\frac{2d}{x} \right] && \text{for } x \gg d \\ &= \frac{2kqd}{x^3}. \end{aligned}$$

Recall that the magnitude of the dipole moment of two point charges is given by $p = qd$. Hence the electric field at point P can be rewritten as

$$\mathbf{E}_P = \frac{2kp}{x^3} \hat{\mathbf{i}} = \frac{\mathbf{p}}{2\pi\epsilon_0 x^3}.$$

- (b) A rod of length, $2L$, lies on the x-axis, centred on origin O . It carries a charge per unit length given by

$$\lambda(x) = \lambda_0 \frac{x}{L}$$

where λ_0 is a constant.

- (i) Find an expression for the electric potential at all points $x > L$ on the x-axis. [5m]

Solution:

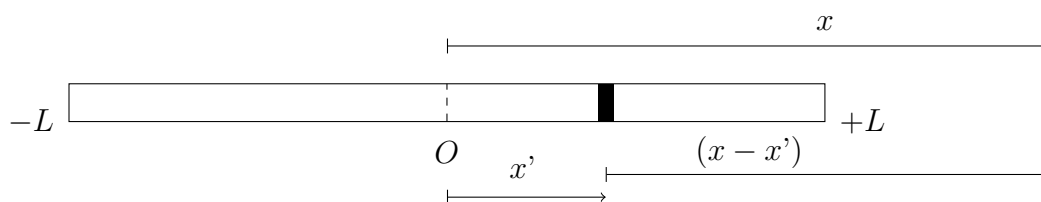


Figure 1: Set-up of the integration.

Consider an infinitesimal element dx' of the rod. The charge on the element is

$$dq = \lambda dx' = \frac{\lambda_0 x'}{L} dx'.$$

The potential at a distance x from the center of the rod is given by the sum of the potentials due to each infinitesimal element of the rod:

$$\begin{aligned} V(x) &= \frac{1}{4\pi\epsilon_0} \int_{-L}^{+L} \frac{dq}{x - x'} = \frac{\lambda_0}{4\pi\epsilon_0 L} \int_{-L}^{+L} \frac{x'}{x - x'} dx' \\ &= \frac{\lambda_0}{4\pi\epsilon_0 L} \int_{-L}^{+L} \left(\frac{x}{x - x'} - 1 \right) dx' \\ &= \frac{\lambda_0}{4\pi\epsilon_0 L} \left[x \ln \left(\frac{x + L}{x - L} \right) - 2L \right]. \end{aligned}$$

As a quick check, we note that

$$\lim_{x \rightarrow \infty} x \ln \left(\frac{x + L}{x - L} \right) = 2L$$

and hence

$$\lim_{x \rightarrow \infty} V(x) = 0 \quad \text{as expected.}$$

- (ii) Hence, find an expression for the electric field at all points $x > L$ on the x-axis. [3m]

Solution:

$$\begin{aligned} \mathbf{E} &= -\frac{\partial V(x)}{\partial x} \hat{\mathbf{i}} = -\frac{\partial}{\partial x} \left(\frac{\lambda_0}{4\pi\epsilon_0 L} \left[x \ln \left(\frac{x+L}{x-L} \right) - 2L \right] \right) \hat{\mathbf{i}} \\ &= -\frac{\lambda_0}{4\pi\epsilon_0 L} \left[\ln \left(\frac{x+L}{x-L} \right) + \frac{x}{x+L} - \frac{x}{x-L} \right] \hat{\mathbf{i}} \\ &= \frac{\lambda_0}{4\pi\epsilon_0 L} \left[\frac{2xL}{x^2 - L^2} - \ln \left(\frac{x+L}{x-L} \right) \right] \hat{\mathbf{i}} \end{aligned}$$

- (iii) Determine the electric dipole moment \mathbf{p} of the rod. Explain very briefly how you obtained your answer. [4m]

Solution: We begin by recognising that the charge density $\lambda(x)$ is an odd function about the origin i.e. $\lambda(x) = \lambda(-x)$. Hence we can divide the rod into infinitesimal dipoles consisting of pairs of infinitesimal elements on opposite sides of the rod. The dipole moment of such an infinitesimal dipole is

$$d\mathbf{p} = 2x'dq \hat{\mathbf{i}} = \frac{2\lambda_0 x'^2}{L} dx' \hat{\mathbf{i}},$$

and hence the total dipole moment of the rod is

$$\mathbf{p} = \int d\mathbf{p} = \int_0^L \frac{2\lambda_0 x'^2}{L} dx' \hat{\mathbf{i}} = \frac{2\lambda_0}{3} L^2 \hat{\mathbf{i}}$$

Problem Seven

- (a) Figure 3 (Refer to the past year question paper) shows a conducting loop lying in the xy -plane. The loop is formed from concentric semicircles of radii a and b . If it carries a current I as shown, find the magnetic field at point P , the common centre. [8m]

Solution:

Recall that the magnetic field due to a current-carrying wire is given by the Biot-Savart law as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{I d\mathbf{l} \times \hat{\mathbf{z}}}{|\mathbf{z}|^2},$$

where $\mathbf{z} \equiv \mathbf{r} - \mathbf{l}$.

The semicircular loops are parameterised by

$$\mathbf{l} = r \cos \theta \hat{\mathbf{i}} + r \sin \theta \hat{\mathbf{j}}, \quad 0 \leq \theta \leq \pi.$$

Hence it follows that

$$d\mathbf{l} = r d\theta \left[-\sin \theta \hat{\mathbf{i}} + \cos \theta \hat{\mathbf{j}} \right] \quad \text{and} \quad \mathbf{z} = -r \cos \theta \hat{\mathbf{i}} - r \sin \theta \hat{\mathbf{j}},$$

and that

$$d\mathbf{l} \times \hat{\mathbf{z}} = rd\theta \left[\sin^2 \theta \hat{\mathbf{k}} + \cos^2 \theta \hat{\mathbf{k}} \right] = rd\theta \hat{\mathbf{k}}.$$

Hence, the magnetic field at the centre due to a semicircular loop becomes

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int_0^\pi \frac{rd\theta \hat{\mathbf{k}}}{r^2} = \frac{\mu_0 I}{4r} \hat{\mathbf{k}}$$

Finally, the resultant magnetic field at point P is given by

$$\frac{\mu_0 I}{4} \left(\frac{1}{a} - \frac{1}{b} \right) \hat{\mathbf{k}},$$

recognising that the current in the outer loop of radius b is flowing in a clockwise direction (hence resulting in an additional negative sign).

- (b) Two concentric, coplanar circular current loops have radii a and $2a$. If the magnetic field is zero at their common centre, how does the current in the outer loop I_{out} compare with the current I_{in} in the inner loop? [4m]

Solution:

The magnetic field at the centre of a circular loop is simply twice that of a semicircular loop:

$$\mathbf{B} = 2 \frac{\mu_0 I}{4r} \hat{\mathbf{k}} = \frac{\mu_0 I}{2r} \hat{\mathbf{k}}$$

The resultant magnetic field at the common centre is then simply

$$\frac{\mu_0 I_{in}}{2a} \hat{\mathbf{k}} + \frac{\mu_0 I_{out}}{4a} \hat{\mathbf{k}} = 0$$

Evidently, the currents must satisfy the relationship

$$I_{out} = -2I_{in},$$

that is, the current in the outer loop has twice the magnitude of the current in the inner loop, and is flowing in an opposite direction from it.

- (c) A solid conducting wire of radius R runs parallel to the z -axis and carries a current density given by

$$\mathbf{J}(r) = J_0 \left(1 - \frac{r}{R}\right) \hat{\mathbf{k}}.$$

Find expressions for

- (a) the total current in the wire. [3m]

Solution:

The total current is simply given by integrating the current density across a cross-sectional area of the wire:

$$\begin{aligned} I_{\text{tot}} &= \int \mathbf{J}(r) \cdot d\mathbf{A} = \int_0^R J_0 \left(1 - \frac{r}{R}\right) (2\pi r) dr = 2\pi J_0 \int_0^R \left(r - \frac{r^2}{R}\right) dr \\ &= 2\pi J_0 \left[\frac{R^2}{2} - \frac{R^3}{3R} \right] = \frac{\pi J_0 R^2}{3} \end{aligned}$$

- (b) the magnetic field strength for $r < R$. [3m]

Solution:

Consider an Amperian loop of radius $r < R$ concentric with the wire. Then by Ampere's law,

$$\begin{aligned} \oint \mathbf{B} \cdot d\mathbf{l} &= B(2\pi r) = \mu_0 I_{\text{enc}} = 2\pi\mu_0 J_0 \int_0^r \left(r' - \frac{r'^2}{R}\right) dr' \\ &= 2\pi\mu_0 J_0 \left[\frac{r^2}{2} - \frac{r^3}{3R} \right]. \end{aligned}$$

Hence,

$$B = \mu_0 J_0 \left[\frac{r}{2} - \frac{r^2}{3R} \right].$$

- (c) the magnetic field strength for $r > R$. [2m]

Solution:

Consider an Amperian loop of radius $r > R$ concentric with the wire. Then by Ampere's law,

$$\begin{aligned} \oint \mathbf{B} \cdot d\mathbf{l} &= B(2\pi r) = \mu_0 I_{\text{enc}} = 2\pi\mu_0 J_0 \int_0^R \left(r' - \frac{r'^2}{R}\right) dr' \\ &= 2\pi\mu_0 J_0 \left[\frac{R^2}{2} - \frac{R^3}{3R} \right] = \mu_0 \frac{\pi J_0 R^2}{3}. \end{aligned}$$

Hence,

$$B = \frac{\mu_0 J_0 R^2}{6r}.$$

Problem Eight

- (a) A square conducting loop of side l and resistance R lies in the xy -plane with its sides parallel to the x - and y -axes. It is being moved with constant velocity $\mathbf{v} = v\hat{\mathbf{i}}$. The magnetic field is given by $\mathbf{B} = B_0\hat{\mathbf{k}}$ for $x < 0$ and $\mathbf{B} = (B_0 + bx)\hat{\mathbf{k}}$ for $x > 0$. At $t = 0$, the trailing side of the loop crosses the y -axis.

- (i) Find an expression for the loop current for times $t \geq 0$. [5m]

Solution:

Faraday's law states that

$$\begin{aligned}\mathcal{E} &= -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A} = \frac{d}{dt} \left[\int_x^{x+l} (B_0 + bx') l dx' \right] \\ &= \frac{d}{dt} \left[B_0 l^2 + \frac{bl}{2} ((x+l)^2 - x^2) \right] \\ &= \frac{d}{dt} \left[B_0 l^2 + \frac{bl}{2} (2xl + l^2) \right] \\ &= bl^2 \frac{dx}{dt} = bl^2 v.\end{aligned}$$

Hence, the current is given by

$$I = \frac{\mathcal{E}}{R} = \frac{bl^2 v}{R}.$$

- (ii) Which way does the current flow as viewed from the positive z -axis? [1m]

Solution:

Since the magnetic flux through the loop is increasing with time, by Lenz's law, the induced emf and hence current flow will be in the direction that produces a magnetic field that opposes the increase in flux – clockwise.

- (b) Consider a pair of vertical long conducting rods a distance l apart and connected at the bottom by a resistance R . A conducting bar of mass m runs horizontally between the rods and can slide freely. The whole apparatus is in a uniform magnetic field \mathbf{B} pointing horizontally and perpendicular to the bar.

- (i) Find the speed of the bar at times $t > 0$ if it is initially released from rest. [6m]

Solution:

The EMF across the bar is given by Faraday's law as

$$\mathcal{E} = \frac{d\Phi_B}{dt} = Bl \frac{dx}{dt} = BLv.$$

The magnetic force acting on the bar is then

$$F_B = BIl = B \left(\frac{\mathcal{E}}{R} \right) l = B \left(\frac{Blv}{R} \right) l = \frac{B^2 l^2 v}{R}.$$

From Newton's Second Law,

$$m \frac{dv}{dt} = mg - F_B = mg - \frac{B^2 l^2 v}{R}.$$

Rearranging and invoking separation of variables,

$$\int_0^t dt = \int_0^v \frac{dv}{g - (B^2 l^2 / mR) v} = -\frac{mR}{B^2 l^2} \ln \left| \frac{g - (B^2 l^2 / mR) v}{g} \right|.$$

Therefore,

$$v(t) = \frac{mRg}{B^2 l^2} \left(1 - \exp \left[-\frac{B^2 l^2}{mR} t \right] \right).$$

- (ii) Hence or otherwise, find v_0 and α . [2m]

Solution:

By direct comparison,

$$v_0 = \frac{mRg}{B^2 l^2} \quad \text{and} \quad \alpha = \frac{B^2 l^2}{mR}.$$

- (c) A conducting disk with radius a , thickness h and resistivity ρ is inside a solenoid of circular cross section. The disk axis coincides with the solenoid axis. The magnetic field in the solenoid at time t is given by $B = bt$. Find expressions for

- (i) the current density in the disk. [2m]

Solution:

Consider a loop of radius $r \leq a$ that is concentric with the disk. Then, by Faraday's law,

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \int_0^r b t (2\pi r') dr' = -\int_0^r b (2\pi r') dr'.$$

Hence,

$$E_\phi (2\pi r) = -b\pi r^2 \quad \Rightarrow \quad E_\phi = -\frac{br}{2}$$

The current density is related to the electric field by $\mathbf{E} = \rho \mathbf{J}$. Therefore the current density in the disk is

$$\mathbf{J}(r) = -\frac{br}{2\rho} \hat{\phi}.$$

- (ii) the power dissipation in the entire disk [4m]

Solution:

Consider a circular current loop of radius r concentric with the disk. The cross-sectional area of the loop is $h dr$. The current flowing in the loop is given by

$$I = \frac{br}{2\rho} h dr,$$

and it encounters a resistance

$$R = \rho \frac{L}{A} = \rho \frac{2\pi r}{h dr}.$$

The power dissipated in this current loop is then

$$dP = I^2 R = \frac{\pi h b^2 r^3}{2\rho} dr.$$

Finally, the total power dissipated in the entire disk is obtained by summing up the power dissipated by each of the current loops:

$$P = \int_0^a \frac{\pi h b^2 r^3}{2\rho} dr = \frac{\pi h b^2 a^4}{8\rho}.$$