NUS Physics Society

Past Year Paper Solutions

PC 1143 : AY 2013 /14

Compiled by:
NUS Physics Society Past Year Solution Team

- Yeo Zhen Yuan
- Ng Kian Fong
- Leong Jin Feng
- Ryan Goh

These solutions were verified by NUS Physics Alumni.

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1. Consider an **isolated** spherical conductor S of radius R carrying a net charge Q.

   (a) Calculate the total work W needed to assemble this charge Q by bringing infinitesimal charges dq from infinity and depositing them on the surface of S. [2]

   Approach

   We first note that the potential at the surface of a uniformly charged sphere with charge q of radius R is given by $V = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{q}{R}$.
   As such the infinitesimal work that is needed to bring infinitesimal charges dq from infinity to the surface of the sphere is:

$$\begin{aligned} dW &= V\ dq = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{q}{R}dq \\ W &= \int_0^Q \frac{q dq}{4\pi\epsilon_0 R} \\ &= \left(\frac{1}{4\pi\epsilon_0}\right)\frac{Q^2}{2R} \end{aligned}$$

   (b) Calculate the electrostatic energy U_E stored in the electric field $\vec{E}$ outside the spherical conductor. [2]

   Approach

   Firstly, use Gauss' law to determine the electric field at all points in space. Using Gauss' Law, we obtain:

$$\vec{E}(\vec{r}) = \begin{cases} 0 & |r| < R \\ \frac{Q}{4\pi\epsilon_0 r^2} & |r| \geq R \end{cases}$$

   We then integrate the electric field energy density $(\frac{1}{2}\epsilon_0 E^2)$ to determine the electrostatic energy stored in the electric field. Note that we can exploit the spherical symmetry of the system and thus replace dV with $4\pi r^2 dr$.

$$\begin{aligned} U_E &= \frac{\epsilon_0}{2}\int_0^\infty \left[\frac{Q}{4\pi\epsilon_0 r^2}\right]^2 4\pi r^2 dr \\ &= \left(\frac{1}{4\pi\epsilon_0}\right)\frac{Q^2}{2R} \end{aligned}$$

(c) Is U_E less than, equal to, or greater than W? Explain very briefly the **physical significance** of your answer. [2]

Approach

U_E is exactly equal to W. This shouldn't come as a surprise as the work done by the external agent in assembling this charge configuration is stored in the electric field produced by this charge configuration.

(d) Hence, or otherwise, find the *capacitance* C of the spherical conductor. [2]

Approach

Recall that the energy stored in the capacitor carrying a charge Q is given by $U = \frac{Q^2}{2C}$. As such, to find the capacitance of this system, we equate this expression of energy with the total energy stored in the electric field, U_E (that we obtained in (b)).

$$\begin{aligned} \frac{Q^2}{2C} &= U_E = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{Q^2}{2R} \\ C &= 4\pi\epsilon_0 R \end{aligned}$$

2. A toroidal coil of square cross section has **inner radius** R and side l as shown in Figure 1. The coil consists of N turns, and carries a current I.

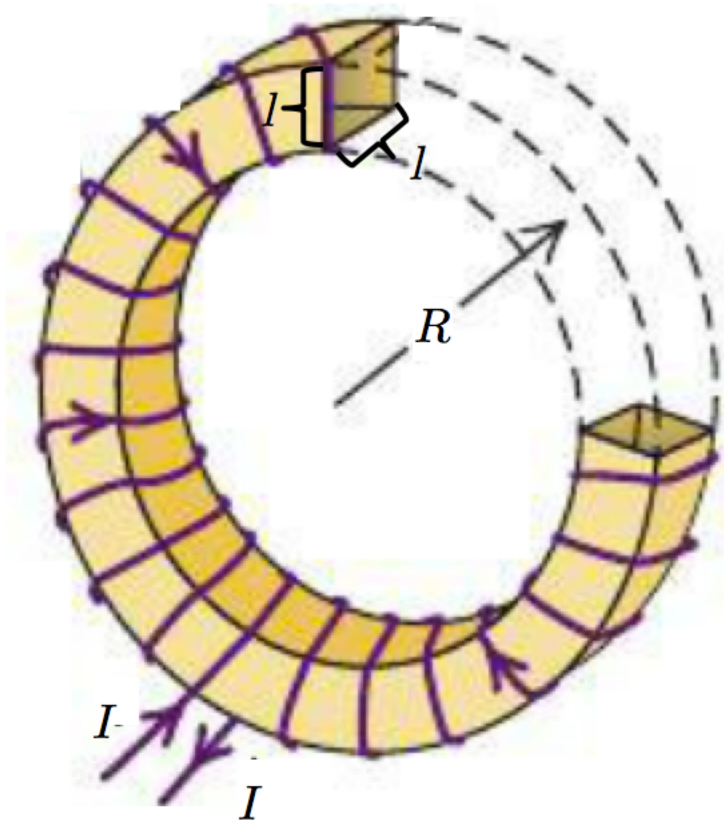

Figure 1

Do NOT assume that magnetic field is uniform across a cross-section

(a) What is the total magnetic energy stored in the toroid? [4]

Approach

We use Ampere's law to determine the magnetic field strength at a point located a distance r from the center of the toroid.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{encl}}$$
$$\vec{B}(\vec{r}) = \frac{\mu_0 N I}{2\pi r}\hat{\phi}$$

Recall that the magnetic field energy density is defined as

$$u_B = \frac{B^2}{2\mu_0}$$

As such, to obtain the total magnetic energy stored in the toroid, we integrate the magnetic field energy density over the entire toroidal volume. Note that the infinitesimal volume we will use for the integration is $dV = 2\pi r l dr$.

$$\begin{aligned} U_B &= \frac{1}{2\mu_0}\int_R^{l+R}\left(\frac{\mu_0 NI}{2\pi r}\right)^2 2\pi r l dr \\ &= \frac{\mu_0 N^2 I^2 l}{4\pi}\ln\left(1+\frac{l}{R}\right) \end{aligned}$$

(b) Hence, or otherwise, determine the *inductance* L of the toroid. [2]

Approach

To find determine the inductance of the toroid, we equate the expression for the total magnetic energy to the energy stored in the inductor, $U = \frac{1}{2}LI^2$.

$$\begin{aligned} \frac{1}{2}LI^2 &= \frac{\mu_0 N^2 I^2 l}{4\pi}\ln\left(1+\frac{l}{R}\right) \\ L &= \frac{\mu_0 N^2 l}{2\pi}\ln\left(1+\frac{l}{R}\right) \end{aligned}$$

(c) Show that L reduces to the inductance of a **long solenoid** when $R >> l$. [2]

Approach

Recall that that the magnetic field inside the long solenoid is uniform. Its magnitude (obtained from Ampere's Law) is $B = \mu_0 \frac{N}{l'} i$ where N is the total number of turns, l' is the total length of the solenoid and i is the magnitude of the current passing through the coils of the solenoid.

The inductance of the long solenoid, with cross-sectional area A is given by:

$$\begin{aligned} \frac{1}{2}LI^2 &= U_B = \frac{\left(\mu_0 \frac{N}{l'} i\right)^2}{2\mu_0} A l' \\ L &= \frac{\mu_0 N^2 A}{l'} \end{aligned}$$

We compare this expression with the one we obtained in (b). In the limit where $R >> l$, The Taylor expansion yields (to first order):

$$\ln\left(1+\frac{l}{R}\right) \approx \frac{l}{R}$$

As such, for the toroidal solenoid,

$$\begin{aligned} L &\approx \frac{\mu_0 N^2 l}{2\pi}\left(\frac{l}{R}\right) \\ &= \frac{\mu_0 N^2 l^2}{2\pi R} \end{aligned}$$

Now, l^2 is 'equivalent' to the cross-sectional area A and $2\pi R$ is the 'effective length' of the toroidal solenoid. Making the appropriate substitutions will show that this reduces to the inductance of the long solenoid.

3. The switch in Figure 2 has been open for a long time. It is closed at time $t = 0$s.

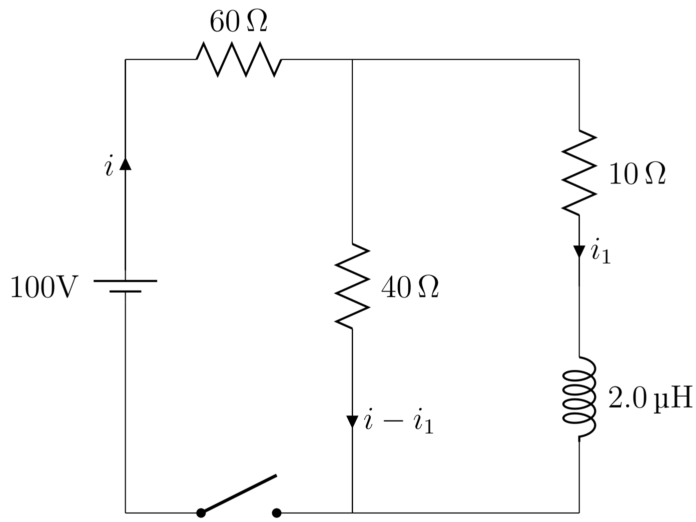

Figure 2

(a) What is the current in the 40 Ω resistor immediately after the switch is closed? [2]

Approach

Immediately after the switch is closed, the current passing through the inductor is 0 (the inductor behaves like a break in the circuit). Thus the circuit reduces to one in which the 40 Ω and 60 Ω resistors are connected to the 100 V battery in series.

$$i = \frac{\mathcal{E}}{R_{\text{effective}}} = \frac{100\text{V}}{40\Omega + 60\Omega} = 1\text{A}$$

(b) Find an expression for the current I through the inductor as a function of time t. [4]

Approach

We apply Kirchoff Loop Rule to this circuit. We define loop 1 to be the path taken when transversing the path starting from the positive terminal of the 100V battery, through the 60 Ω and the 40 Ω resistor and back to the negative terminal of the 100V battery.

We define loop 2 to be the path taken when transversing the path starting from the 10 Ω resistor through the 2 μH inductor and upwards through the 40 Ω resistor (going against the 'proposed' direction of current).

$$\begin{aligned} \text{Loop 1} : 100 - 60i - 40(i - i_1) &= 0 \\ \text{Loop 2} : -10i_1 - (2.0 \times 10^{-6})\frac{di_1}{dt} + 40(i - i_1) &= 0 \end{aligned}$$

To arrive at an expression for $i_1(t)$, we need to find an expression for i using the equation obtained for loop 1 and substitute that expression for i into the equation obtained for loop 2.

$$\begin{aligned}
i &= \frac{100 + 40i_1}{100} \\
50i_1 + (2 \times 10^{-6})\frac{di_1}{dt} &= 40\left(\frac{100 + 40i_1}{100}\right) \\
\int_0^{i_1(t)} \frac{di_1'}{40 - 34i_1'} &= \int_0^t \frac{1}{2 \times 10^{-6}} dt' \\
i_1(t) &= \frac{20}{17}\left[1 - \exp(-17 \times 10^6 t)\right]
\end{aligned}$$

(c) What is the current in the 10 Ω resistor after the switch has been closed for a long time? [2]

Approach

To arrive at the answer for this part, take the limit of the expression obtained in (b) as $t \to \infty$.

$$i = \frac{20}{17}\mathrm{A}$$

In this case the voltage dropped across the inductor vanishes.

4. Consider the series RLC circuit in Figure 3.

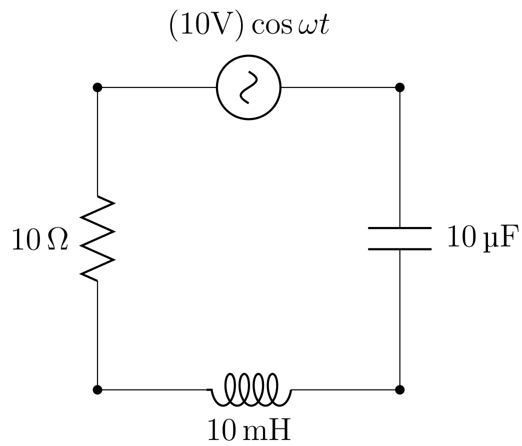

Figure 3

(a) Find the *impedance* of the circuit. Express your answer in terms of ω. [2]

Approach

Recall the expression for the impedance of an AC circuit:

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

Substitute in the known values: $R = 10\Omega$, $L = 10\text{mH}$ and $C = 10\mu\text{F}$.

$$Z = \sqrt{100 + \left(\omega(10^{-2}) - \frac{1}{\omega(10^{-5})}\right)^2}$$

(b) What is the *resonance frequency*, in both rad/s and Hz? [2]

Approach

Recall that the resonance frequency is obtained when inductive reactance equals the inductive impedance.

$$\begin{aligned} \omega L &= \frac{1}{\omega C} \\ \omega &= \sqrt{\frac{1}{LC}} \end{aligned}$$

Substituting in the known values, we obtain:

$$\begin{aligned} \omega &= 3162\text{rad/s} \\ f &= \frac{\omega}{2\pi} = 503\text{Hz} \end{aligned}$$

(c) Find V_R and V_L at resonance. [2]

Approach

First, we need to find the current amplitude at resonance.

$$I_0 = \frac{V_0}{Z} = \frac{10}{\sqrt{100}} = 1.0\text{A}$$

We then use this value of the current amplitude to determine V_L and V_R. Recall that $V_L = I_0\omega L$ and $V_R = I_0 R$. Substitute the value of ω which we obtained in (b) to determine V_R and V_L.

$$\begin{aligned} V_R &= 10\text{V} \\ V_L &= 31.62\text{V} \end{aligned}$$

(d) How can V_L be larger than 10 V? Explain. [2]

Approach

In a series alternating current circuit, the voltage amplitudes across the inductor and capacitor are not in phase with the current. As such, V_L can be larger than 10 V as long as the phasor sum of all the voltage amplitudes equal 10 V.

5. At one instant, the electric field $\vec{E}$ and magnetic field $\vec{B}$ at one point of an *electromagnetic wave* are

$$\vec{E} = (200\hat{\imath} + 300\hat{\jmath} - 50\hat{k})\text{V/m}$$

and

$$\vec{B} = B_0(7.30\hat{\imath} - 7.30\hat{\jmath} + \alpha\hat{k})\mu\text{T}$$

(a) What are the values of α and B_0? [4]

Approach

There are two unknowns which we need to find, namely α and B_0. This means that we require two sets of equations.

The first equation relates the magnitude of the electric field to that of the magnetic field.

$$|\vec{E}| = c|\vec{B}|$$

The second equation is the mathematical statement of the orthogonality relationship between the electric field and the magnetic field.

$$\vec{E} \cdot \vec{B} = 0$$

We can determine the value of α using the second equation.

$$\begin{aligned} 200(7.30) + 300(-7.30) - 50\alpha &= 0 \\ \alpha &= -14.6 \end{aligned}$$

We then use this value of α to obtain the value of B_0 using the first equation. Note that the magnitude of the magnetic field is expressed in terms of micro-Teslas (μT).

$$\begin{aligned}\sqrt{200^2 + 300^2 + (-50)^2} &= B_0(3.0 \times 10^8)(10^{-6})\sqrt{7.3^2 + (-7.3)^2 + (-14.6)^2} \\ B_0 &= 0.06786\end{aligned}$$

(b) What is the *Poynting Vector* at this time and position? [4]

Approach

Recall the definition of the *Poynting Vector*.

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

To obtain the value of the *Poynting Vector*, we substitute the values of α and B_0 into the expression for the magnetic field and evaluate the cross product. Note that the resulting answer should yield a vector.

$$\begin{aligned}\vec{S} &= \frac{10^{-6} B_0}{\mu_0}\left(-4745\hat{\imath} + 2555\hat{\jmath} - 3650\hat{k}\right) \\ &= 0.0540\left(-4745\hat{\imath} + 2555\hat{\jmath} - 3650\hat{k}\right)\end{aligned}$$

6. A pair of **equal** but **opposite** charges $+q$ and $-q$, lies on the x axis at $x = -a$ and $x = +a$ respectively, as shown in Figure 4.

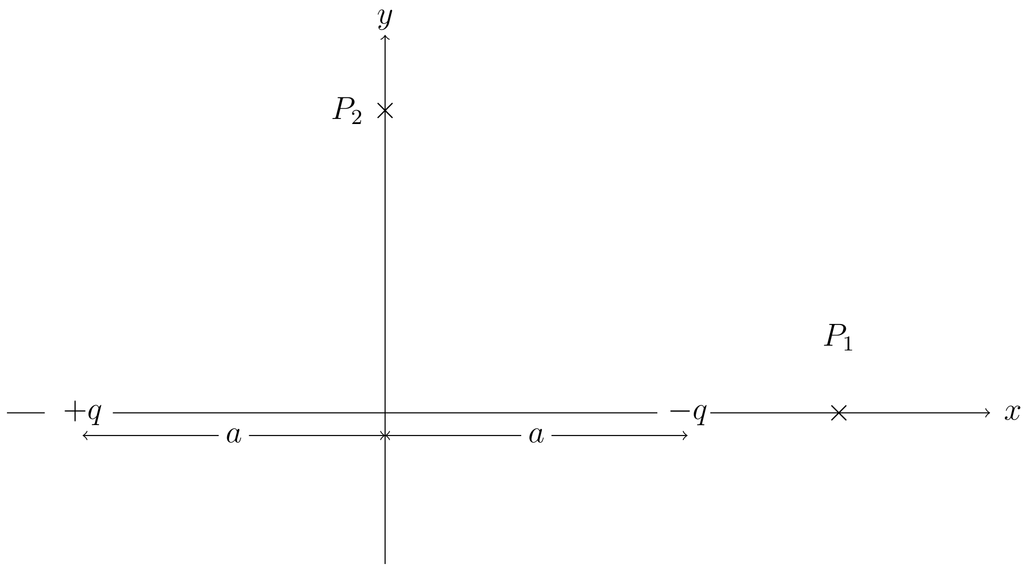

Figure 4

(a) Find the electric potential V_1 at point $P_1(x, 0)$ on the x axis. [2]

Approach

Recall the definition of the electric potential,

$$V = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{q}{|\vec{r} - \vec{r'}|}$$

Hence, at point P_1, the electric potential is given by:

$$\begin{aligned} V_1 &= \left(\frac{q}{4\pi\epsilon_0}\right)\left[\frac{1}{x+a} - \frac{1}{x-a}\right] \\ &= \left(\frac{q}{4\pi\epsilon_0}\right)\left[\frac{-2a}{(x-a)(x+a)}\right] \end{aligned}$$

(b) Write down the mathematical relationship between the electrostatic field $\vec{E}$ and the potential V at a point in space. [2]

Approach

Recall that $\vec{E} = -\nabla V$.

(c) Hence, find the electric field $\vec{E}$ at point P_1. [2]

Approach

This questions stems from the implicit 'hint' given in (b). To find the electric field $\vec{E}$ at P, we take the negative of the gradient of the potential function at P_1.

$$\begin{aligned}\vec{E} &= -\nabla\left\{\left(\frac{q}{4\pi\epsilon_0}\right)\left[\frac{1}{x+a} - \frac{1}{x-a}\right]\right\} \\ &= \left(\frac{q}{4\pi\epsilon_0}\right)\left[\frac{1}{(x+a)^2} - \frac{1}{(x-a)^2}\right]\hat{\imath}\end{aligned}$$

(d) Find the electric field $\vec{E_2}$ at point $P_2(0, y)$ on the y axis. [4]

Approach

Note that at point P_2, the resultant electric field is purely in the $+x$ direction.

$$\vec{E_2} = 2\left(\frac{q}{4\pi\epsilon_0}\right)\left(\frac{1}{y^2+a^2}\right)\cos(\theta)\hat{\imath}$$

where

$$\cos(\theta) = \frac{a}{\sqrt{y^2+a^2}}$$

Therefore,

$$\vec{E_2} = \frac{2qa}{4\pi\epsilon_0\left(y^2+a^2\right)^{\frac{3}{2}}}\hat{\imath}$$

(e) What is the electric potential V_2 at P_2.? Does your answer contradict that in (d)? Explain briefly. [4]

Approach

The electric potential V_2 at P_2 is 0. At first glance, this seems to contradict our answer we proposed in (d) as $\vec{E} = -\nabla V$ and thus it is very tempting to conclude that since the $\nabla(0) = 0$, a contradiction exists.

However, this is clearly wrong. To see how we can arrive at our answer in (f), let's consider the potential function over the xy plane, $V(x, y)$.

$$V(x,y) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x+a)^2+y^2}} - \frac{1}{\sqrt{(x-a)^2+y^2}}\right]$$

As such, the expression for the electric field everywhere on the xy plane is given by:

$$\begin{aligned}\vec{E}(x,y) &= -\nabla V(x,y) \\ &= -\frac{q}{4\pi\epsilon_0}\left\{\left[\frac{-(x+a)}{[(x+a)^2+y^2]^{\frac{3}{2}}} + \frac{(x-a)}{[(x-a)^2+y^2]^{\frac{3}{2}}}\right]\hat{\imath}\right. \\ &\quad \left. + \left[\frac{-y}{[(x+a)^2+y^2]^{\frac{3}{2}}} + \frac{y}{[(x-a)^2+y^2]^{\frac{3}{2}}}\right]\hat{\jmath}\right\}\end{aligned}$$

When we substitute $x = 0, y = y'$, we get:

$$\vec{E}(0, y') = \frac{2qa}{4\pi\epsilon_0 \left(y'^2 + a^2\right)^{\frac{3}{2}}}\hat{i}$$

as expected.

(f) Find $\vec{E}_1$ where $x >> a$, and $\vec{E}_2$ where $y >> a$. Express your answers in terms of the **electric dipole moment** of the system of two charges. What is common to your answers? [6]

Approach

Recall our expression for $\vec{E}_1$.

$$\begin{aligned}\vec{E}_1 &= \left(\frac{q}{4\pi\epsilon_0}\right)\left[\frac{1}{(x+a)^2} - \frac{1}{(x-a)^2}\right]\hat{i} \\ &= \left(\frac{q}{4\pi\epsilon_0}\right)\left[\frac{-4qxa}{(x+a)^2(x-a)^2}\right]\hat{i}\end{aligned}$$

The electric dipole moment for this system of two charges is (recall that the direction vector points from the negative charge to the positive charge):

$$\vec{p} = -2qa\hat{i}$$

As such, when $x >> a$, $(x+a)^2 \approx x^2$ and $(x-a)^2 \approx x^2$. Therefore,

$$\begin{aligned}\vec{E}_1 &\approx \left(\frac{1}{4\pi\epsilon_0}\right)\left[\frac{-4qxa}{x^4}\right] \\ &= \left(\frac{2}{4\pi\epsilon_0 x^3}\right)\vec{p}\end{aligned}$$

We repeat the same procedure for E_2. Recall that:

$$\vec{E}_2 = \frac{2qa}{4\pi\epsilon_0 \left(y^2 + a^2\right)^{\frac{3}{2}}}\hat{i}$$

As such, when $y >> a$, $(y^2 + a^2)^{\frac{3}{2}} \approx y^3$. Therefore,

$$\begin{aligned}\vec{E}_2 &\approx \frac{2qa}{4\pi\epsilon_0 y^3}\hat{i} \\ &= -\left(\frac{1}{4\pi\epsilon_0 y^3}\right)\vec{p}\end{aligned}$$

The common theme present in both our answers is that in the limit of large distances, the dipole field goes down in $\frac{1}{r^3}$ as opposed to $\frac{1}{r^2}$ for a point charge.

7. A straight section of wire of length L carries a current I, as shown in Figure 5.

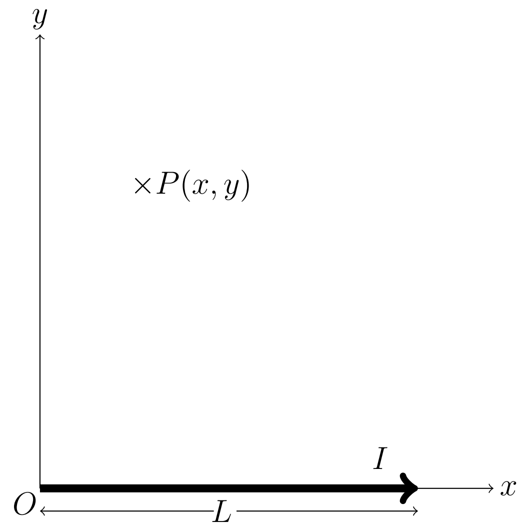

Figure 5

(a) Show that that the magnetic field $\vec{B}$ associated with this segment at point P is given by

$$\vec{B} = \frac{\mu_0 I}{4\pi y}\left[\frac{x}{\sqrt{x^2+y^2}} - \frac{x-L}{\sqrt{(x-L)^2+y^2}}\right]\hat{k}$$

[6]

Approach

We use the Law of Biot and Savart for this part of the question. Firstly, lets establish some parameters:

$$\begin{aligned}
\vec{r}-\vec{r'} &= (x-x')\hat{\imath} + y\hat{\jmath} \\
|\vec{r}-\vec{r'}| &= \sqrt{(x-x')^2+y^2} \\
d\vec{l} &= dx'\hat{\imath} \\
d\vec{l}\times(\vec{r}-\vec{r'}) &= ydx'\hat{k}
\end{aligned}$$

Recall the Law of Biot Savart:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\times(\vec{r}-\vec{r'})}{\left|\vec{r}-\vec{r'}\right|^3}$$

We now insert the relevant parameters into the integral, yielding:

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_0^L \frac{ydx\hat{k}}{[(x-x')^2+y^2]^{\frac{3}{2}}}$$

Evaluate the integral to get the expression stated in the question.

$$\vec{B} = \frac{\mu_0 I}{4\pi y}\left[\frac{x}{\sqrt{x^2+y^2}} - \frac{x-L}{\sqrt{(x-L)^2+y^2}}\right]\hat{k}$$

(b) Hence, find $\vec{B}$ when P is very close to the current-carrying wire. Explain how you could apply Ampere's law to determine $\vec{B}$ in this case. [4]

Approach

Now, when P is very close to the current-carrying wire, we note that $y << x$. In this limit,

$$\begin{aligned} \frac{x}{\sqrt{x^2+y^2}} &\approx 1 \\ \frac{x-L}{\sqrt{(x-L)^2+y^2}} &\approx \frac{x-L}{|x-L|} \end{aligned}$$

Do note that since $x < L$, $|x-L| = L-x$. Therefore,

$$\vec{B} \approx \frac{\mu_0 I}{4\pi y}\left[1-(-1)\right]\hat{k} = \frac{\mu_0 I}{2\pi y}\hat{k}$$

We can also obtain the above result through the use of Ampere's Law. Recall:

$$\int \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{encl}}$$

To apply Ampere's law, we just draw a circular Amperian loop enclosing the wire. We then take B out of the integral and note that $\int d\vec{l} = 2\pi r$ over a circular loop.

$$\begin{aligned} B(2\pi r) &= \mu_0 I \\ \vec{B} &= \frac{\mu_0 I}{2\pi r}\hat{\phi} \end{aligned}$$

(this is because wire can be approximated to be infinite.)

(c) Find the magnetic field $\vec{B}$ at point Q in Figure 6. [6]

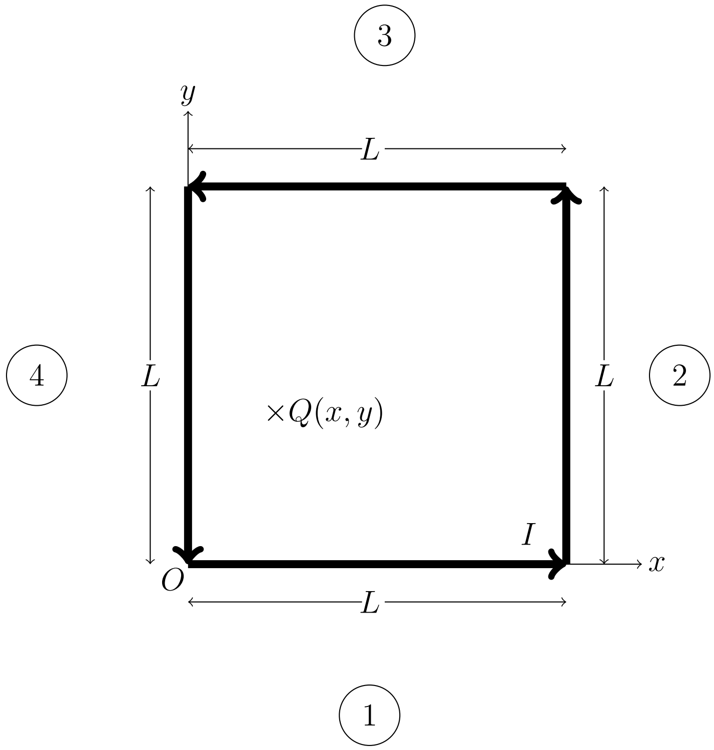

Figure 6

Approach

We will analyze the magnetic field due to each current segment individually as labelled in Figure 6.

We have obtained the result for the magnetic field due to current segment 1. Let's call this magnetic field expression $\vec{B}_1$.

$$\vec{B}_1 = \frac{\mu_0 I}{4\pi y}\left[\frac{x}{\sqrt{x^2+y^2}} - \frac{x-L}{\sqrt{(x-L)^2+y^2}}\right]\hat{k}$$

Observe the symmetry - to obtain the magnetic field due to current segment 3, we just need to replace every y in the expression for $\vec{B}_1$ with $L - y$.

$$\vec{B}_3 = \frac{\mu_0 I}{4\pi(L-y)}\left[\frac{x}{\sqrt{x^2+(L-y)^2}} - \frac{x-L}{\sqrt{(x-L)^2+(L-y)^2}}\right]\hat{k}$$

We observe the symmetry again (this time for current segment 4). The magnetic field expression for current segment 4 can be obtained by making the following substitution in the magnetic field expression due to current segment 1.

$$\begin{aligned} x &\rightarrow y \\ y &\rightarrow x \end{aligned}$$

As a result, we will obtain:

$$\vec{B}_4 = \frac{\mu_0 I}{4\pi x}\left[\frac{y}{\sqrt{y^2+x^2}} - \frac{y-L}{\sqrt{(y-L)^2+x^2}}\right]\hat{k}$$

Finally, to obtain the magnetic field due to current segment 2, we simply replace every x in the expression for $\vec{B}_4$ with $L-x$.

$$\vec{B}_2 = \frac{\mu_0 I}{4\pi(L-x)}\left[\frac{y}{\sqrt{y^2+(L-x)^2}} - \frac{y-L}{\sqrt{(y-L)^2+(L-x)^2}}\right]\hat{k}$$

To get the expression for the total magnetic field, we simply sum up the expressions $\vec{B}_1, \vec{B}_2\vec{B}_3$ and $\vec{B}_4$.

$$\vec{B} = \frac{\mu_0 I}{4\pi}\left[\frac{\frac{x}{y}+\frac{y}{x}}{\sqrt{x^2+y^2}} + \frac{\frac{y}{L-x}-\frac{x-L}{y}}{\sqrt{(x-L)^2+y^2}} + \frac{\frac{x}{L-y}-\frac{y-L}{x}}{\sqrt{x^2+(y-L)^2}} - \frac{\frac{x-L}{L-y}+\frac{y-L}{L-x}}{\sqrt{(x-L)^2+(y-L)^2}}\right]\hat{k}$$

(d) Is the magnetic field $\vec{B}$ at R greater or less than at the centre of the square? Justify your answer. [4]

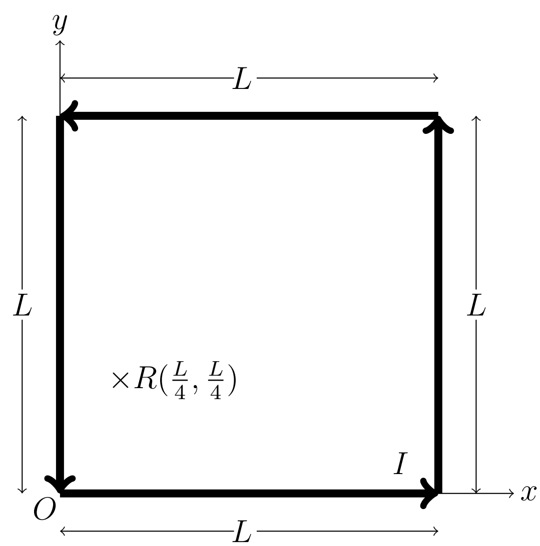

Figure 7

Approach

To determine the magnetic field at the center of the square and at the point R, we just need to substitute the known coordinates into our expression we obtained in (c).

At the center of the square, $x = y = \frac{L}{2}$. Therefore,

$$\begin{aligned}
\frac{x}{y} = \frac{y}{x} = \frac{y}{L-x} = \frac{x}{L-y} &= 1 \\
\frac{x-L}{y} = \frac{y-L}{x} = \frac{x-L}{L-y} = \frac{y-L}{L-x} &= -1 \\
\sqrt{x^2+y^2} = \sqrt{(x-L)^2+y^2} = \sqrt{x^2+(y-L)^2} = \sqrt{(x-L)^2+(y-L)^2} &= \frac{L}{2}\sqrt{2}
\end{aligned}$$

As such,

$$\begin{aligned}
\vec{B}_{\text{center}} &= \frac{\mu_0 I}{4\pi\left(\frac{L}{2}\sqrt{2}\right)}\{[1+1]+[1-(-1)]+[1-(-1)]-[(-1)+(-1)]\}\,\hat{k} \\
&= \left(2\sqrt{2}\right)\left(\frac{\mu_0 I}{\pi L}\right)\hat{k} \\
&= 2.82843\left(\frac{\mu_0 I}{\pi L}\right)\hat{k}
\end{aligned}$$

We now proceed to find the expression for the magnetic field at the point R $(\frac{L}{4}, \frac{L}{4})$. Note that at R,

$$\begin{aligned}
\frac{x}{y} = \frac{y}{x} &= 1 \\
\frac{y}{L-x} = \frac{x}{L-y} &= \frac{1}{3} \\
\frac{x-L}{y} = \frac{y-L}{x} &= -3 \\
\frac{x-L}{L-y} = \frac{y-L}{L-x} &= -1 \\
\sqrt{x^2+y^2} &= \frac{L}{4}\sqrt{2} \\
\sqrt{(x-L)^2+y^2} = \sqrt{x^2+(y-L)^2} &= \frac{L}{4}\sqrt{10} \\
\sqrt{(x-L)^2+(y-L)^2} &= \frac{3L}{4}\sqrt{2}
\end{aligned}$$

As such,

$$\begin{aligned}
\vec{B_R} &= \frac{\mu_0 I}{4\pi}\left[\frac{2}{\frac{L}{4}\sqrt{2}} + \frac{\frac{10}{3}}{\frac{L}{4}\sqrt{10}} + \frac{\frac{10}{3}}{\frac{L}{4}\sqrt{10}} + \frac{2}{\frac{3L}{4}\sqrt{2}}\right]\hat{k} \\
&= \left[\frac{4}{3}\sqrt{2} + \frac{20}{3\sqrt{10}}\right]\frac{\mu_0 I}{\pi L}\hat{k} \\
&= \approx 3.9938\left(\frac{\mu_0 I}{\pi L}\right)\hat{k}
\end{aligned}$$

As such, it is clear that $\vec{B_R}$ (off-center) is larger than $\vec{B}_{\text{center}}$.

8. Consider a conducting ring of radius R, uniform circular cross-sectional area A, and resistivity ρ, lying with its plane perpendicular to a uniform magnetic field $\vec{B}$ as shown in Figure 8.

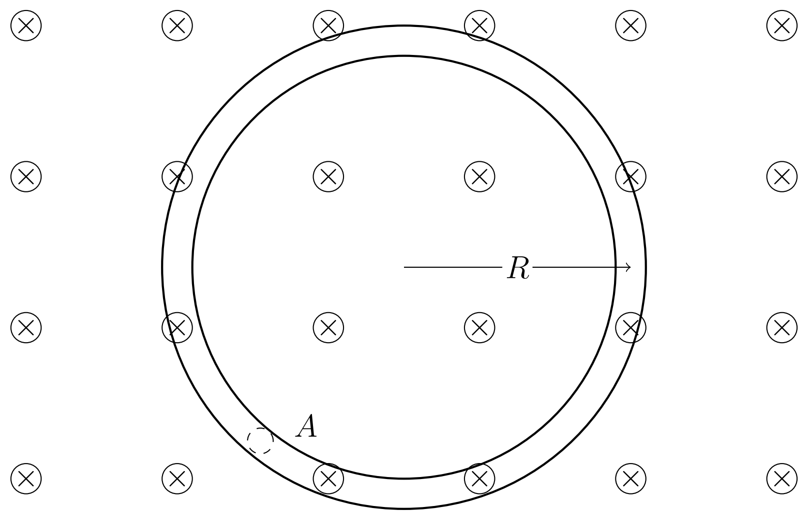

Figure 8

(a) Suppose the magnetic field B at time t is given by

$$B = B_0[1 - \exp(-bt)]$$

with b and B_0 positive constants.

i. Find an expression for the induced current density in the ring as a function of time t. [6]

Approach

Recall Faraday's Law:

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

To apply Faraday's Law for this case, we draw an imaginary circular loop of radius r. Since the electric field is tangential, we can bring it out of the integral. The magnetic flus through this circular area is $\Phi_B = \pi R^2 B_0[1 - \exp(-bt)]$. We thus obtain:

$$\begin{aligned} E(2\pi R) &= -\pi R^2 B_0 b \exp(-bt) \\ E &= -\frac{RB_0 b}{2} \exp(-bt) \end{aligned}$$

$\vec{A}$ and $\vec{B}$ are parallel. Hence +ve circulation about $\vec{A}$ is clockwise. Recall that $|\vec{E}| = \frac{1}{\rho}|\vec{J}|$ where $|\vec{J}|$ is the magnitude of the current density. Thus the induced current density as a function of time t is:

$$J = -\frac{RB_0 b}{2\rho} \exp(-bt) \quad ,\text{(anti-clockwise)}$$

ii. Hence, find by integrating the induced current over time the total charge that moves around the ring as B increases from zero to B_0. [4]

Approach

The induced current that flows is given by $I = |\vec{J}|A$, where A is the cross sectional area. Note also that it takes an infinite amount of time for B to increase from zero to B_0.

The total charge that moves is thus given by:

$$\begin{aligned} Q &= \int_0^\infty -\frac{RB_0 b}{2\rho} A \exp(-bt)dt \\ &= \frac{RB_0}{2\rho} A \end{aligned}$$

(b) Figure 9 shows a generator consisting of a conducting rod of length R that rotates with angular speed ω about a central axis Q while making contact with the conducting ring.

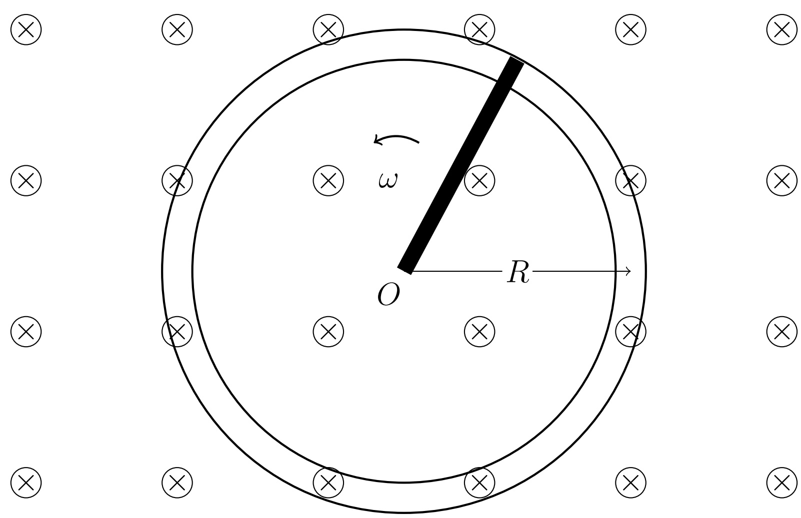

Figure 9

Suppose the magnetic field $B = B_0$ is now constant.

i. Find an expression for the electric field in the rod as a function of the distance r from the central axis, when the conduction electrons in the rod are in equilibrium. [2]

Approach

The question states that the conduction electrons in the rod are in equilibrium. This means that the Lorentz force equals to 0.

$$\vec{F} = \left(q\vec{E} + q\vec{v} \times \vec{B}\right) = 0$$

Note that the $|v| = r\omega$ and that $\vec{v} \times \vec{B}$ is directed towards O. We rearrange to solve for $\vec{E}$.

$$\vec{E}(\vec{r}) = B\omega r\hat{r}$$

where $\hat{r}$ is the unit vector pointing radially outwards from O.

ii. Hence, or otherwise, find an expression for the emf induced in this generator. Specify if the emf induced is directed away or towards the central axis. [3]

Approach

Recall that the induced emf, $\mathcal{E}$ is defined as:

$$\mathcal{E} = \int \vec{E} \cdot d\vec{l}$$

We substitute our expression obtained for $\vec{E}$ into the equation. We will get:

$$\mathcal{E} = \frac{1}{2}B\omega r^2$$

The induced emf, $\mathcal{E}$, is directed towards the central axis.

iii. Now, wires from the axis and ring carry power to a load. If the induced current in the circuit is I_0, find an expression for the rate of work done by an external agent to maintain the angular speed of the rod at ω. [3]

Approach

The rate of work done by the external agent to maintain the angular speed of the rod at ω must be equal to the rate of power dissipation of the load. We define P to be the rate of work done by the external agent.

$$\begin{aligned} P &= \mathcal{E}I_0 \\ &= \frac{1}{2}BI_0\omega r^2 \end{aligned}$$

iv. Hence, or otherwise, find an expression for the load resistance. [2]

Approach

This rate of dissipation must be equal to the Joule heating due to the load resistance. We define R to be the load resistance.

As such,

$$\begin{aligned} \frac{1}{2}BI_0\omega r^2 &= I_0^2 R \\ R &= \frac{B\omega r^2}{2I_0} \end{aligned}$$