Suggested Solutions for Modern Physics AY2007-08 Semester 2

NUS Physics Society

Part I

1. (a) Total energy is the sum of the kinetic energy and the rest energy of a particle. The inclusion of the rest energy in the calculation of the total energy of a particle implies that the mass of a particle is equivalent to energy.

   (b) Total energy, $E = \gamma mc^2$, while relativistic momentum, $p = \gamma mu$

$$\begin{aligned} \text{Hence, } E^2 - p^2c^2 &= (\gamma mc^2)^2 - (\gamma mu)^2c^2 \\ &= \gamma^2(mc^2)[1 - u^2/c^2] \\ &= mc^2 \end{aligned}$$

   (c) From the proven equation in part (1)(b) above, when $m << E$, $m \approx 0$

$$E^2 - p^2c^2 = 0$$
$$\text{Hence, } E = pc$$

   The de Broglie's wavelength of a particle can be found through the equation $p = h/\lambda$. Hence,

$$E = pc = \frac{hc}{\lambda}$$

   which is the equation for the energy of a photon.

2. (a) In a harmonic oscillator, the potential energy of the system, $U = \frac{1}{2}kx^2 = \frac{1}{2}\omega^2x^2m$. Hence, the Schrodinger equation can be written as $\frac{\delta^2\psi}{\delta x^2} = \frac{2m}{\hbar^2}(\frac{1}{2}kx^2 - E)\psi(x)$. One of the solutions for this system is $\psi = Ce^{-m\omega x^2/(2\hbar)}$, with C as the normalization constant, and $E = \frac{1}{2}\hbar\omega$. For the diagrams of the wave functions and the probability densities, please refer to the next page.

   Note: Solving the Schrodinger equation is beyond the scope of this course, but students should at least remember the solution given in the lecture notes.

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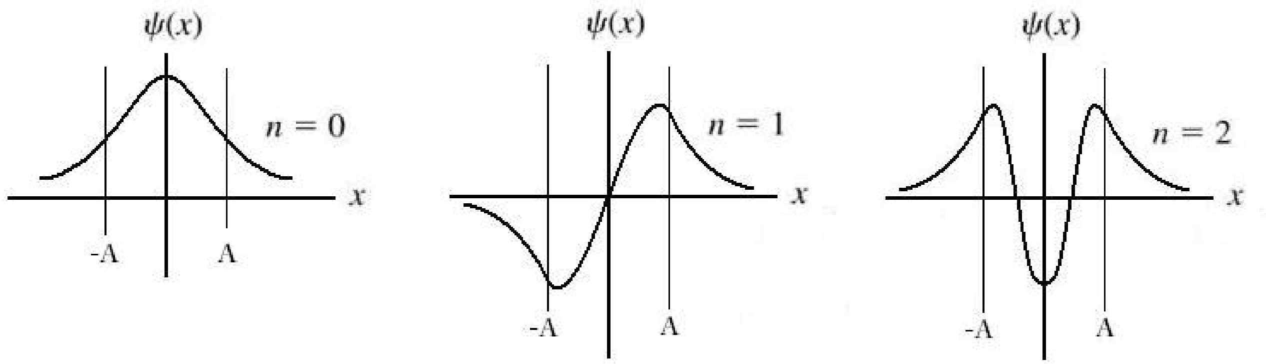

Figure 1: Wavefunctions of the first 3 energy states, for Question 2(a)

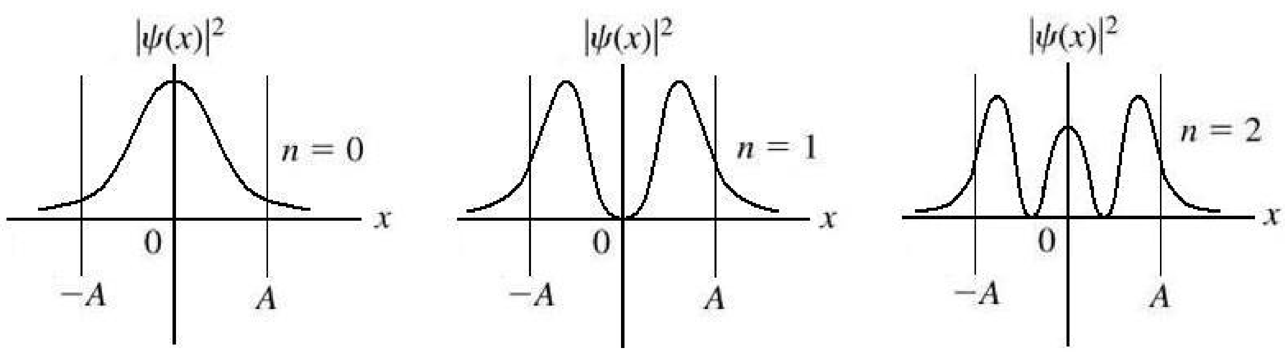

Figure 2: Probability densities of the first 3 energy states, for Question 2(a)

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(b) In the classical model, the atom would spend more time at the amplitudes if the system is simple harmonic, while in the quantum case, there are certain regions where there is zero probability of finding the atom, as seen in the probability density diagrams. For n=1, there is zero probability of finding the particle at $x = 0$, which contradicts the classical argument. Also, classically, the particle should not be in regions where $|x| > A$, where A is the amplitude. However, in the quantum case, there is a non-zero probability of finding the particle in regions outside the amplitude.

3. From Heisenberg Uncertainty Principle, $\Delta x \Delta p \geq \hbar/2$. On the other hand, the kinetic energy, KE of a confined particle can be expressed as KE$= \frac{1}{2}mv^2 = \frac{p^2}{2m}$.

   In this case, the neutron is confined within the nucleus. The value of Δx should not be more than the diameter of the nucleus d, that is, $\Delta \leq d$. To find the minimum energy, we can set $\Delta x = d$. Hence, Δp can be expressed as $\Delta p \geq \frac{\hbar}{2d} = \frac{h}{4\pi d}$.

   To find the minimum kinetic energy of the neutron, the momentum p must at least be comparable to the uncertainty in terms of magnitude. Hence, we can set $p = \Delta p$ to find the minimum energy. Therefore,

$$KE = \frac{p^2}{2m} = \frac{h^2}{2m(4\pi d)^2}$$

Substituting the values into the equation, $KE = \dfrac{(6.63 \times 10^{-34})^2}{2(1.67 \times 10^{-27})(4\pi \times 10^{-14})^2}$J

Hence, $KE = 8.33 \times 10^{-15}\text{J} = 0.05\text{MeV}$

4. (a) The binding energy of a nucleus is the energy that must be given into a system of bound nucleons to separate them into the individual protons and neutrons. Vice versa, the binding energy of a nucleus can also be thought of as the energy released when nucleons join together. Overall, the binding energy of a nucleus is the result of the mass defect of the nucleus compared to its constituents.

   (b) Let the binding energy of carbon 12 be E_B, the mass of proton to be M_p, the mass of neutron to be M_n and the mass of a carbon-12 nucleus to be M_{C-12}

$$\begin{aligned} E_B &= [6M_p + 6M_n - M_{C-12}]c^2 \\ &= [6(938.28) + 6(939.57) - 11177.9]\text{MeV} \\ &= 89.2\text{MeV} \end{aligned}$$

5. The decay follows the equation $K^+ = \pi^+ + \pi^+ + \pi^-$. Using the conservation of energy, total energy

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of K^+ = total energy of decay products. (Note: total energy = rest energy + kinetic energy)

$$\begin{aligned}\text{Hence, energy of } K^+ &= [68.6 + 80.8 + 75.5 + 3(139.6)]\text{MeV} \\ &= 643.7\text{MeV} \\ &= 150\text{MeV} + (\text{mass of } K^+)c^2 \\ \text{Hence, mass of } K^+ &= 493.7\text{MeV}/c^2\end{aligned}$$

Part II

6. (a)

$$\begin{aligned}\Delta x' &= \gamma(\Delta x - v\Delta t) \\ \Delta t' &= \gamma(\Delta t - v\Delta x/c^2) \\ \text{Hence, } \frac{\Delta x'}{\Delta t'} &= \frac{\Delta x - v\Delta t}{\Delta t - v\Delta x/c^2} \\ &= \frac{\Delta x/\Delta t - v}{1 - \left(\frac{v}{c^2}\right)\left(\frac{\Delta x}{\Delta t}\right)}\end{aligned}$$

Given that $\Delta x/\Delta t = u_x$ and $\Delta x'/\Delta t' = u'_x$,

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2}$$

(b)

$$\begin{aligned}u'_x &= \frac{u_x - v}{1 - vu_x/c^2} \\ &= \frac{07c - 0.75c}{1 - (0.7)(0.75)} \\ &= -0.118c\end{aligned}$$

The captain of the defender spaceship will believe that he is overtaking the enemy spaceship at 0.118c.

(c) The photon torpedo will strike the defender spaceship at the speed of light, c. The defender spaceship cannot take evasive action and avoid the photon torpedo, because from the reference frame of the defender spaceship, the photon torpedo is also approaching the ship at the speed

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of light, c. This is in accordance with Einstein's Special Theory of Relativity, which states that in inertail frames, the speed of light is constant, independent of the velocity of the source or the observer.

7. (a) i.

$$y(x) = A\sin(n\pi x/L) = \psi$$

In this case, where the particle is trapped in a rectangular box with infinite walls, the potential energy, U is given as below:

$$\begin{aligned} U(x) &= 0, \text{ for } 0 \leq x \leq L \\ U(x) &= \infty \text{ for } x < 0 \text{ and } x > L \end{aligned}$$

The expression for the energy levels can be derived through the use of Schrodingers equation,

$$\begin{aligned} \frac{\delta^2\psi}{\delta x^2} &= -\frac{2m}{\hbar^2}(E-U)\psi \\ \text{Hence, } \frac{\delta^2}{\delta x^2}[A\sin(n\pi x/L)] &= -\frac{2m}{\hbar^2}(E)[A\sin(n\pi x/L)] \\ \left(\frac{n\pi}{L}\right)^2 &= \frac{2m}{\hbar}(E) \\ E &= \frac{h^2n^2}{8mL^2} \end{aligned}$$

ii. For ground state, $n = 1$, while for the second excited state, $n = 3$. Hence, the difference between the second excited state and the ground state is

$$E_3 - E_1 = \frac{h^2}{8mL^2}(3^2 - 1^2) = \frac{h^2}{mL^2}$$

(b) The electron is initially in the first excited state, that is, $n = 2$. After dropping into the ground state, $n = 1$. Given that the width of the box, $L = 0.50$nm,

$$E_2 - E_1 = \frac{h^2}{8mL^2}(2^2 - 1^2) = \frac{3h^2}{8mL^2} = \frac{hc}{\lambda}$$

Hence, $\lambda = 91.66$nm.

(c) i. The wave functions for the first three energy states are shown in the diagram on the next page. The dotted line represents the wave function for $\psi = 0$.

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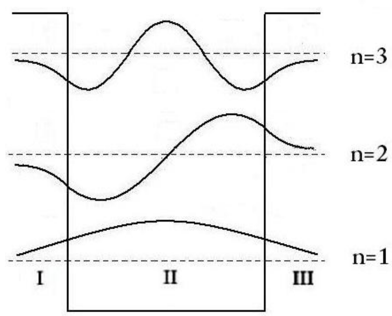

Figure 3: Wave functions of the first 3 energy states, for Question 7(c)(i)

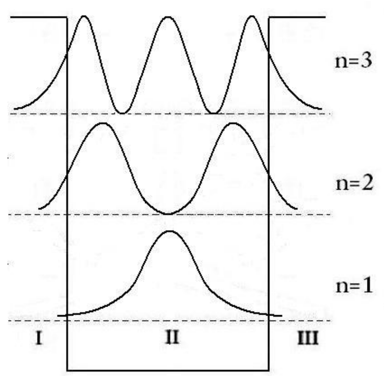

Figure 4: Probability densities of the first 3 energy states, for Question 7(c)(i)

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ii. The probability density for the first three energy states are shown in the diagram on the next page. The dotted line corresponds to $|\psi|^2 = 0$.

For an infinite well, the wave functions are zero at $x = 0$ and $x = L$. However, for a finite well, the wave function is not zero at those points, and hence, the wavelengths, λ, of the sinusoidal parts of the wave functions are longer. With a larger λ, the momentum, $p = h/\lambda$ is smaller. This corresponds to a lower energy for each energy level of a finite well compared to the corresponding energy level for an infinite well.

Besides that, while an infinite well has infinitely many bound states, there are only a finite amount of bound states for a finite well. However, since the wavefunction is not zero at the points $x = 0$ and $x = L$ in a finite well, the wavefunction penetrates the well and there is a non-zero probability of finding the particle outside the boundaries of the well. This is known as quantum tunneling, and is more prominent in finite wells.

8. (a) Alpha decay is the process in which an alpha particle or a Helium nucleus is emitted.The alpha particle consists of 2 protons and 2 neutrons, and this decay usually originate from nclei of heavy elements. Classically, the alpha particle would not be able to escape from the nucleus, and alpha decay cannot occur. However, alpha decay can be explained using quantum mechanics, where the whole procress is represented by the tunneling through a potential barrier.

Beta decay involves the emission of beta particles, which can either be electrons, β^-, or positrons, β^-. For beta decay to occur, there is a conversion of neutron into a proton, or vice versa, in the nucleus. The beta particle is created during the decay. When an electron is emitted, there is the emission of an anti neutrino as well, while the emission of a positron will be accompanied by a neutrino. Neutrinos and anti neutrinos are brought into to explain the wide range of kinetic energies of beta particles. During beta decay, the daughter nucleus has the same nucleon number, but the atomic number will increase by 1.

Gamma decay is the decay process in which gamma rays are emitted. Gamma rays are high energy photons, which are emitted when the daughter nuclues resulting from an alpha or beta decay returns to the ground state from an excited state. Gamma decay does not change the number of protons or nucleons in the nucleus; only the internal arrangement of nucleons is changed. Gamma decay can also be stimulated by colliding the nucleus with another fast moving particle, a move used in nuclear reactions. The photon emitted usually is high in energy, and have high power of penetration.

(b) ^{14}C is the radioactive isotope, which exists in the atmosphere in the form of carbon dioxide ($^{14}CO_2$), used in carbon dating. At any time, the ratio of ^{14}C with respect to ^{12}C in the atmosphere at a constant ratio, that is, at 1.3×10^{-12}.

Living organisms will have the same ratio of ^{14}C with respect to ^{12}C, due to the intake of

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carbon dioxide into their systems. However, once the organism dies, the intake of carbon dioxide stops and the ^{14}C in the carcass decays accordingly. The activity of the ^{14}C in the dead organism at the current time, R, can then be compared to the original activity, R_0 to obtain the age of the specimen, t. The relationship between R and R_0 are as below:

$$R = R_0 e^{-\lambda t}$$

The value of λ is known experimentally, while the value of R can be measured with a detector.

(c) The activity of the ^{14}C in the sample will be used to calculate the age of the sample. Firstly, given that the half life of the ^{14}C is 5730 years, we can calculate the value of the decay constant, λ.

$$\begin{aligned} \lambda &= \frac{0.693}{T_{1/2}} = \frac{0.693}{(5370\text{years})(3.16 \times 10^7\text{sec/year})} \\ &= 4.084 \times 10^{-12}\text{s}^{-1} \end{aligned}$$

The original activity, R_0 of the ^{14}C is given to be 31 decays per minute, that is, the activity of the living organism. Using the equation $R = R_0 e^{-\lambda t}$,

$$\begin{aligned} \frac{26\text{counts}}{1\text{min} \times 60\text{sec/minute}} &= \frac{31\text{counts}}{1\text{min} \times 60\text{sec/minute}} \times e^{-4.084 \times 10^{-12}\text{s}^{-1}(t)} \\ 4.333 \times 10^{-1} &= 5.167 \times 10^{-1} e^{-4.084 \times 10^{-12}\text{s}^{-1}(t)} \\ -1.760 \times 10^{-1} &= -4.084 \times 10^{-12}\text{s}^{-1}(t) \\ \text{Hence, } t &= 4.310 \times 10^{10}\text{s} = 1.364 \times 10^3\text{years} \end{aligned}$$

(d) The surface activity of the Sun will affect the production of ^{14}C in the atmosphere. High energy photons from the increase in solar activity will increase the number of ^{14}C in the atmosphere, and subsequently, cause a rise in the ratio of ^{14}C to ^{12}C. This higher ratio will not affect the activity in the sample as much as it will affect the activity of ^{14}C in the living sample, R_0, which was used as a comparison. The higher value of R_0 will cause the value of t that was determined to be smaller than the actual value. Hence, the axe handle will be dated to be more recent than its actual age.

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