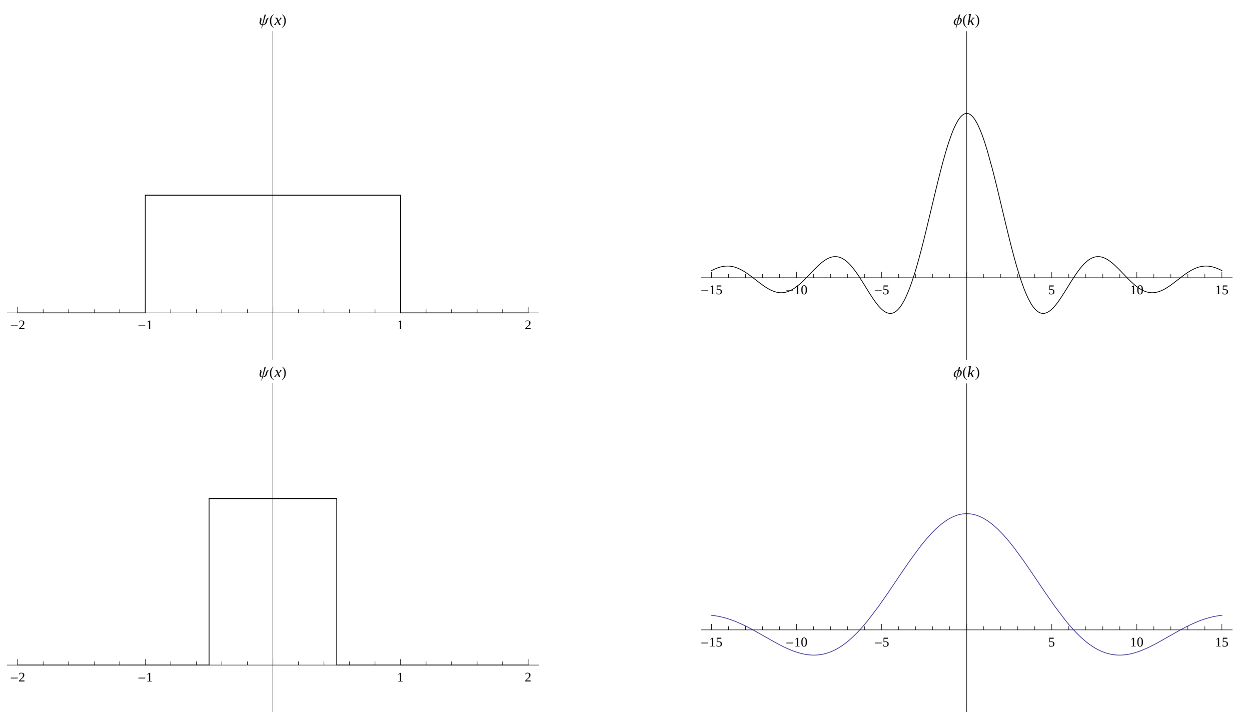
PC1144: Physics IV (AY2011/2012 sem 2)
Suggested solutions

As the question sheet was not available after the exam, the questions listed below may not be an exact reproduction of the questions in the exam.

The answers provided below for qualitative questions should not be treated as definitive, as other answers may be acceptable. In addition, it may not be necessary to state all the points listed for each question — depending on the number of marks assigned to that question, it may be possible to obtain full credit by only stating several of the listed points.

1a. Shown below on the left are two position wavefunctions, with the Fourier transform of the first wavefunction shown beside it on the right. Sketch the Fourier transform of the second wavefunction in the space provided. (2 marks)

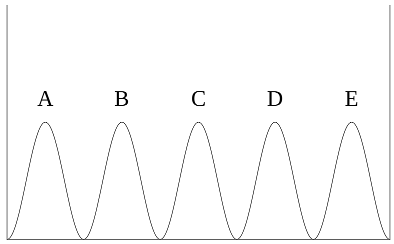


(Answer is shown in blue. The main point to note is that when the square-pulse position wavefunction halves in width and increases in height, the momentum wavefunction (i.e. the Fourier transform) doubles in width and decreases in height.)

1b. Name the principle behind your answer to part (a). (2 marks)

The Heisenberg uncertainty principle.

1c. Shown below is a sketch of $|\psi|^2$ for a particle in an infinite square well. With respect to the points A, B, C, D, E, describe how the behaviour of the particle compares to the classical case as it moves from point A to E. (4 marks)



- Unlike the classical case, the particle does not have a well-defined location. Instead, the probability of locating the particle in a given region is described by the probability density function $|\psi|^2$.
- The particle is most likely to be found in the vicinity of points A, B, C, D, E.
- There is zero probability of finding the particle at the points where the wavefunction is zero, i.e. the points midway between points A, B, C, D, E.
- This is in contrast to the classical case, in which the particle is equally likely to be found at all points within the well (assuming an ensemble of systems with random starting times).
- Unlike the classical case, the particle cannot be thought of as "passing through" the points midway between points A, B, C, D, E.
- In both the classical and quantum cases, the expectation value (mean) of the particle's position is the midpoint of the box.

2a. Comment on the decay possibilities and properties of the following reactions. (4 marks)

i. $n \to p + e^- + \bar{v}_e$

- Charge, lepton number and baryon number are conserved, therefore this decay should be possible.
- Flavour is not conserved, as a down quark has been changed to an up quark. Hence this reaction must proceed via the weak interaction.
- This is the basic reaction behind β^- decay.
- As the mass of the reactants is greater than the mass of the products, it is possible for this reaction to occur without external energy input.

ii. $p \to n + e^+ + v_e$

- Charge, lepton number and baryon number are conserved, therefore this decay should be possible.
- Flavour is not conserved, as an up quark has been changed to a down quark. Hence this reaction must proceed via the weak interaction.
- This is the basic reaction behind β^+ decay.
- As the mass of the reactants is less than the mass of the products, this reaction can only proceed with a source of energy input (such as some initial kinetic energy of the proton, or a change in binding energy of a nucleus).

2b. What were some of the problems with the Bohr theory? (3 marks)

- The Bohr model predicts an incorrect angular momentum for the ground state of the hydrogen atom (it predicts non-zero angular momentum for all states, whereas the Schrödinger theory indicates that the ground state possesses zero angular momentum).
- It is generally unable to predict the spectra of multiple-electron atoms.
- It is unable to predict rates of electronic transitions and hence the relative intensities of spectral lines.
- It does not properly account for spectral line splitting due to spin-orbit coupling (between the orbital angular momentum and electron spin and/or nuclear spin).
- It violates the Heisenberg uncertainty principle as it proposes definite orbits and angular momenta for the electrons.
- It was an apparently ad-hoc mixture of classical theories and wave-particle duality, which was felt to be unsatisfactory.

2c. Is the Schrödinger equation compatible with Einstein's theory of special relativity? Why or why not? (1 mark)

No. Einstein's theory of special relativity indicates that space and time should be treated on equal footing, whereas the Schrödinger equation is second-order in space but first-order in time; indicating that it does not treat the coordinates equally.

3a. Write down the Lorentz transformations in terms of the coordinates x, y, z and ct. (Express your answer in terms of $\beta = \frac{v}{c}$ and $\gamma = (1-\beta^2)^{-1/2}$.) (4 marks)

$$
\begin{aligned}
x' &= \gamma(x - \beta(ct)) \\
y' &= y \\
z' &= z \\
ct' &= \gamma(ct - \beta x)
\end{aligned}
$$

3b. Show that under the substitution $\phi = \tanh^{-1}\beta$, the Lorentz transformations can be represented by the following matrix equation: (4 marks)

$$
\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\phi & 0 & 0 & -\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}
$$

(Hint: $e^{\phi} = \sqrt{\dfrac{1+\beta}{1-\beta}}$, $e^{-\phi} = \sqrt{\dfrac{1-\beta}{1+\beta}}$)

$$
\begin{aligned}
\cosh\phi = \frac{e^{\phi}+e^{-\phi}}{2} &= \frac{1}{2}\left(\sqrt{\frac{1+\beta}{1-\beta}} + \sqrt{\frac{1-\beta}{1+\beta}}\right) \\
&= \frac{1}{2}\left(\frac{(1+\beta)+(1-\beta)}{1-\beta^2}\right) \\
&= \frac{1}{1-\beta^2} \\
&= \gamma
\end{aligned}
\qquad
\begin{aligned}
\sinh\phi = \frac{e^{\phi}-e^{-\phi}}{2} &= \frac{1}{2}\left(\sqrt{\frac{1+\beta}{1-\beta}} - \sqrt{\frac{1-\beta}{1+\beta}}\right) \\
&= \frac{1}{2}\left(\frac{(1+\beta)-(1-\beta)}{1-\beta^2}\right) \\
&= \frac{\beta}{1-\beta^2} \\
&= \gamma\beta
\end{aligned}
$$

Therefore, $$\begin{pmatrix} \cosh\phi & 0 & 0 & -\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} \gamma(x-\beta(ct)) \\ y \\ z \\ \gamma(ct - \beta x) \end{pmatrix}$$
which corresponds to the Lorentz transformations as shown in part (a).

Alternative: Making use of the hyperbolic relations $\cosh^2\phi - \sinh^2\phi = 1$ and $\tanh\phi = \frac{\sinh\phi}{\cosh\phi}$,

$$
\cosh^2\phi - \sinh^2\phi = 1 \implies
\begin{aligned}
1 - \tanh^2\phi &= \frac{1}{\cosh^2\phi} \\
\cosh^2\phi &= \frac{1}{1-\tanh^2\phi} \\
\cosh^2\phi &= \frac{1}{1-\beta^2} \\
\cosh\phi &= \gamma
\end{aligned}
\qquad \text{and} \qquad
\begin{aligned}
\frac{1}{\tanh^2\phi} - 1 &= \frac{1}{\sinh^2\phi} \\
\sinh^2\phi &= \frac{\tanh^2\phi}{1-\tanh^2\phi} \\
\sinh^2\phi &= \frac{\beta^2}{1-\beta^2} \\
\sinh\phi &= \gamma\beta
\end{aligned}
$$

4a. The rate of radioactive decay of nuclei is proportional to the number of remaining nuclei, i.e. $\frac{dN}{dt} = -\lambda N$ where λ is a positive constant. Solve this equation for the number of nuclei $N(t)$ as a function of time. (2 marks)

$$\frac{dN}{dt} = -\lambda N \implies \int_{N_0}^{N} \frac{1}{N}\, dN = -\lambda \int_0^t dt \quad \text{where } N_0 \text{ is the number of nuclei at } t = 0.$$

$$\ln \frac{N}{N_0} = -\lambda t$$

$$N(t) = N_0 e^{-\lambda t}$$

4b. The activity $A(t)$ of a sample is the number of decays per second, i.e. $A(t) = \left|\frac{dN}{dt}\right|$. From your answer to part (a), express the activity as a function of time. (2 marks)

$$A(t) = \left|\frac{dN}{dt}\right| = \left|\frac{d}{dt}\left(N_0 e^{-\lambda t}\right)\right| = \lambda N_0 e^{-\lambda t} \quad \text{since } \lambda \text{ and } N_0 \text{ are positive constants}$$

4c. Let the number of nuclei that decay in the small time interval $(t, t+dt)$ be $f(t)\,dt$. Show that $f(t)\,dt = \lambda N_0 e^{-\lambda t}\,dt$. (1 mark)

At a particular time t, the number of nuclei remaining is $N(t) = N_0 e^{-\lambda t}$.

Therefore, the number of nuclei that have decayed between t and $t + dt$ is:

$$f(t)\,dt = |N(t+dt) - N(t)| = \left|\frac{dN}{dt}\right| dt = \lambda N_0 e^{-\lambda t}\,dt.$$

4d. Find the mean lifetime τ of each radioactive nucleus in terms of λ. (2 marks)

Based on part (c), the proportion $P(t)\,dt$ of nuclei with lifetime t is $P(t)\,dt = \frac{f(t)\,dt}{N_0} = \lambda e^{-\lambda t}\,dt$.

$$\text{The mean lifetime is hence } \tau = \int_0^\infty tP(t)\,dt = \int_0^\infty t\left(\lambda e^{-\lambda t}\right)\,dt$$

$$= \left[-te^{-\lambda t}\right]_0^\infty - \int_0^\infty -e^{-\lambda t}\,dt$$

$$= \lim_{t\to\infty}\left(-te^{-\lambda t}\right) - 0 + \left[-\frac{1}{\lambda}e^{-\lambda t}\right]_0^\infty$$

$$= \frac{1}{\lambda}$$

(Note: $\lim_{t\to\infty}\left(-te^{-\lambda t}\right) = \lim_{t\to\infty}\left(-\frac{t}{e^{\lambda t}}\right) = \lim_{t\to\infty}\left(-\frac{1}{\lambda e^{\lambda t}}\right) = 0$ by l'Hôpital's rule.)

4e. Show that the half-life $T_{1/2}$ and mean lifetime τ are related by $T_{1/2} = \tau \ln 2$. (1 mark)

By definition, the half-life is the time taken for half the nuclei to decay. Therefore, we must have

$$N(t) = N_0\left(\frac{1}{2}\right)^{\frac{t}{T_{1/2}}} = N_0\left(e^{\ln\frac{1}{2}}\right)^{\frac{t}{T_{1/2}}} = N_0 e^{-\frac{\ln 2}{T_{1/2}}t}$$

By comparison to the equation $N(t) = N_0 e^{-\lambda t}$, it can be seen that $\lambda = \frac{\ln 2}{T_{1/2}}$. Combined with the result from part (d), we hence have $T_{1/2} = \frac{\ln 2}{\lambda} = \tau \ln 2$.

5a. Write some brief notes on each of the following topics:

i. Dark matter and dark energy (2 marks)

- Dark matter refers to matter that does not interact significantly with electromagnetic radiation except through its gravitational effects.
- One reason for proposing its existence was to account for the rotational speed of stars near the edges of galaxies, which were higher than that expected from the visible matter alone.
- It is classified into hot and cold dark matter, referring to particles moving close to/significantly slower than the speed of light respectively.
- Dark energy was proposed to account for the fact that the acceleration of the universe is not consistent with the observed/calculated amount of matter/dark matter.
- Dark energy acts to increase the acceleration of the universe.
- The universe is estimated to be composed of about 23% dark matter and 73% dark energy.

ii. Nuclear fission and nuclear fusion (2 marks)

- Fission refers to large nuclei splitting apart into two or more smaller nuclei and neutrons, releasing energy in the process.
- Fission occurs naturally in various elements, usually those of high atomic mass.
- Fusion refers to several small nuclei combining to form a larger nucleus, releasing energy in the process.
- Fusion requires significantly higher temperatures and pressures to achieve than nuclear fission, in order to overcome the inter-nuclei repulsion. It occurs naturally in stellar cores.
- Fusion typically releases much more energy per unit mass of reactants than fission.

iii. The liquid-drop and shell models (4 marks)

- The liquid-drop model treats the nucleus as a drop of incompressible fluid.
- The semi-empirical mass formula is derived taking the liquid-drop model into account, by introducing terms accounting for number of protons, surface area of the drop, and so on.
- The shell model of the nucleus treats the nucleons as particles in a potential well, similar to the electron-in-Coulomb-potential for the Schrödinger picture of the hydrogen atom.
- By applying the Schrödinger equation and solving for the wavefunctions of the nucleons in the potential well, the energy levels of the nucleus can be calculated and found to possess a configuration similar to the "shell structure" for electrons.
- The results indicate that there should be "magic numbers" of nucleons which result in highly stable nuclei.

5b. List all matter and force particles in the Standard Model. (2 marks)

The four field particles: Gravitons, photons, gluons, $W^+/W^-/Z$ bosons
The six leptons: Electrons, electron neutrinos, muons, muon neutrinos, tauons, tau neutrinos
The six flavours of quarks: Up, down, strange, charmed, top, bottom
and their corresponding antiparticles.

5c. What are the difficulties in unifying gravitation, as described by general relativity, with the Standard Model? (2 marks)

General relativity describes gravitation as a curvature of space-time, whereas the Standard Model describes forces as being mediated by particles. It is difficult to reconcile the geometric description of gravity in general relativity with the idea of particles as force carriers.

6a. State the one-dimensional time-dependent Schrödinger equation and the main ansatz involved in solving it. (2 marks)

$$i\hbar\frac{d\Psi}{dt} = -\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + V\Psi \qquad \text{(where } \Psi \text{ is a function of } x \text{ and } t.\text{)}$$

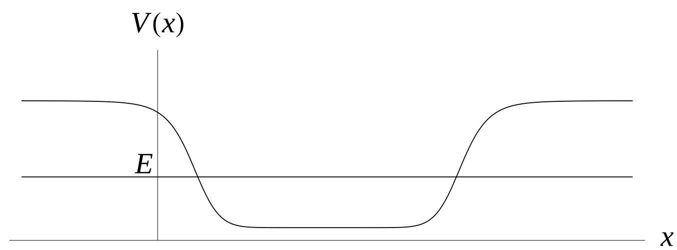
The main ansatz involved in solving it is $\Psi(x,t) = \psi(x)\phi(t)$, which allows the above equation (which involves both position and time coordinates) to be separated into two equations, each in terms of only either x or t alone (provided V is time-independent). The time component can then be solved to yield $\phi(t) = e^{-i\frac{E}{\hbar}t}$, while the position component depends on the potential V.

6b. Show that the one-dimensional time-independent Schrödinger equation can be expressed in the following form (3 marks):

$$\frac{d^2\psi}{dx^2} = \frac{2m(V-E)}{\hbar^2}\psi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \implies -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E-V)\psi$$
$$\implies \frac{d^2\psi}{dx^2} = \frac{2m(V-E)}{\hbar^2}\psi$$

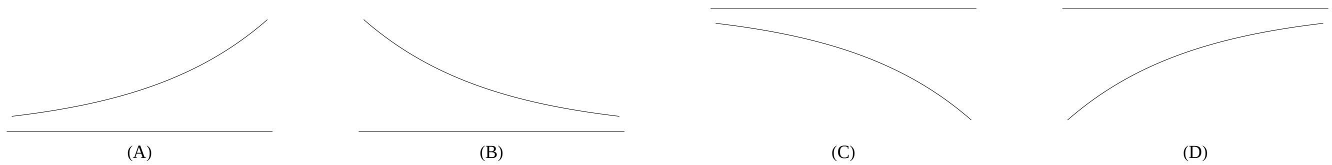
6c. Shown below is the graph of a one-dimensional potential well $V(x)$, along with the energy E of a particle in the well. Answer the following questions based on this potential.



i. What are the conditions for the wavefunction ψ and its first derivative? (1 mark)

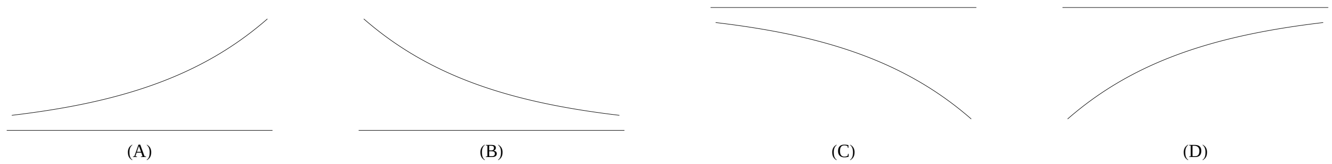
Since the potential is finite at all points in the diagram, both ψ and $\frac{d\psi}{dx}$ must be finite, single-valued and continuous.

ii. If ψ is positive, which option(s) below represent possible wavefunction(s) for the particle in the well? (1 mark)



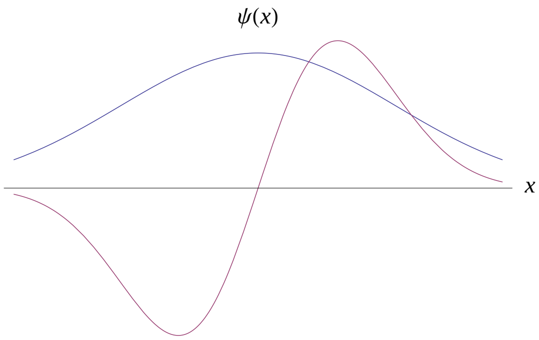
Question appears to be ambiguous, since the above options all do not represent valid wavefunctions within the region shown in the graph of $V(x)$. However, if we assume that the region shown in the above options refers to the region on the right of the graph (specifically, the region on the right in which $V(x) > E$), then option (B) is valid.

iii. If ψ is negative, which option(s) below represent possible wavefunction(s) for the particle in the well? (1 mark)



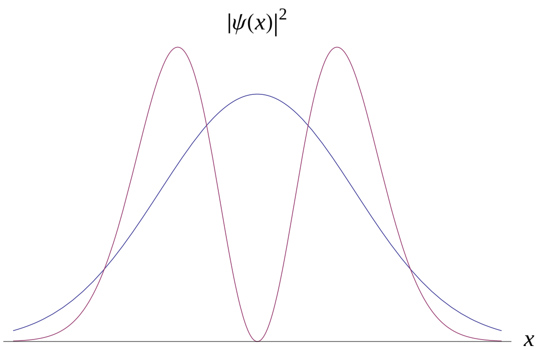
Question appears to be ambiguous, since the above options all do not represent valid wavefunctions within the region shown in the graph of $V(x)$. However, if we assume that the region shown in the above options refers to the region on the right of the graph (specifically, the region on the right in which $V(x) > E$), then option (D) is valid.

iv. Sketch a possible wavefunction for the particle in the well. (2 marks)



Two possible answers are shown above. The points where $(V(x) - E)$ changes sign correspond to inflection points of ψ in the above graph (because as seen from part (b), $\frac{d^2\psi}{dx^2}$ changes sign when either $(V(x) - E)$ or ψ changes sign).

v. Sketch a possible probability density function for the particle in the well. (2 marks)



Two possible answers are shown above.

7. The scattering of electromagnetic radiation from a charged particle can be analysed by a differential equation of the following form, where ω, ω_0, τ and F are constants:

$$\frac{dx}{dt} - \left(i\omega_0 - \frac{\tau\omega_0^2}{2}\right) x = Fe^{i\omega t}$$

7a. Show that $x = Ae^{i\lambda t}$, where A is an arbitrary constant and $\lambda = \omega_0 + \frac{i\tau\omega_0^2}{2}$, is a solution to the above differential equation when $F = 0$. (2 marks)

Substituting the expression $x = Ae^{i\lambda t}$ into the above differential equation,

$$\begin{aligned}
\text{LHS} = \frac{dx}{dt} - \left(i\omega_0 - \frac{\tau\omega_0^2}{2}\right) x &= i\lambda\left(Ae^{i\lambda t}\right) - \left(i\omega_0 - \frac{\tau\omega_0^2}{2}\right)\left(Ae^{i\lambda t}\right) \\
&= \left(i\left(\omega_0 + \frac{i\tau\omega_0^2}{2}\right) - i\omega_0 + \frac{\tau\omega_0^2}{2}\right)\left(Ae^{i\lambda t}\right) \\
&= 0 \\
&= \text{RHS} \quad \text{when } F = 0.
\end{aligned}$$

7b. In the case where F can be non-zero, the ansatz $x = \phi e^{i\omega t}$ is a possible solution to the differential equation. Find ϕ. (3 marks)

Substituting the ansatz into the given differential equation, we require:

$$i\omega\left(\phi e^{i\omega t}\right) - \left(i\omega_0 - \frac{\tau\omega_0^2}{2}\right)\left(\phi e^{i\omega t}\right) = Fe^{i\omega t}$$

$$\left(i\omega - i\omega_0 + \frac{\tau\omega_0^2}{2}\right)\phi = F, \text{ cancelling out the common factor } e^{i\omega t}$$

$$\phi = \frac{F}{i(\omega - \omega_0) + \frac{\tau\omega_0^2}{2}}$$

7c. The intensity $I(\omega)$ of the re-emitted radiation is proportional to $|\phi|^2$ (i.e. $\phi\phi^*$). Show that $\frac{I(\omega)}{I(\omega_0)} = \frac{\frac{\tau^2\omega_0^4}{4}}{(\omega-\omega_0)^2 + \frac{\tau^2\omega_0^4}{4}}$. (3 marks)

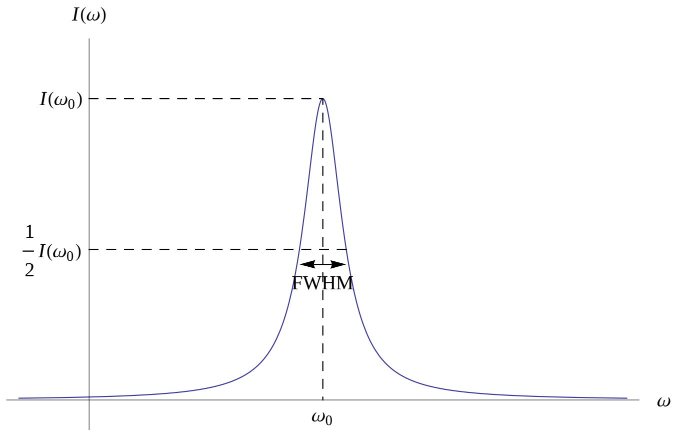
Since $I(\omega) \propto |\phi|^2$, we have

$$\begin{aligned}
I(\omega) = k\,|\phi|^2 &= k\left|\frac{F}{i(\omega - \omega_0) + \frac{\tau\omega_0^2}{2}}\right|^2 \\
&= k\frac{|F|^2}{\left|i(\omega - \omega_0) + \frac{\tau\omega_0^2}{2}\right|^2} \\
&= \frac{k\,|F|^2}{(\omega-\omega_0)^2 + \frac{\tau^2\omega_0^4}{4}} \quad \text{assuming } \omega,\ \omega_0, \text{ and } \tau \text{ are real}
\end{aligned}$$

Therefore, $I(\omega_0) = \dfrac{k\left|F\right|^2}{(\omega_0 - \omega_0)^2 + \frac{\tau^2\omega_0^4}{4}} = \dfrac{k\left|F\right|^2}{\frac{\tau^2\omega_0^4}{4}}$

And hence $\dfrac{I(\omega)}{I(\omega_0)} = \dfrac{k\left|F\right|^2 / \left((\omega - \omega_0)^2 + \frac{\tau^2\omega_0^4}{4}\right)}{k\left|F\right|^2 / \left(\frac{\tau^2\omega_0^4}{4}\right)} = \dfrac{\frac{\tau^2\omega_0^4}{4}}{(\omega - \omega_0)^2 + \frac{\tau^2\omega_0^4}{4}}$

7d. Shown below is a graph of $I(\omega)$ against ω. The *full width at half maximum* (FWHM) of a peak in some function f is a measure of the peak width. It is calculated by finding the maximum value f_{peak} at the top of the peak, then finding the width of the graph at the value $\frac{1}{2}f_{peak}$, as shown in the diagram below. Based on the result of part (c), find the FWHM of the intensity peak in terms of τ and ω_0. (2 marks)



(Verifying that $I(\omega_0)$ is the maximum intensity): $\dfrac{dI(\omega)}{d\omega} = 0 \implies -\dfrac{2(\omega - \omega_0)I(\omega_0)\frac{\tau^2\omega_0^4}{4}}{\left((\omega - \omega_0)^2 + \frac{\tau^2\omega_0^4}{4}\right)^2} = 0$

$$\implies \omega = \omega_0$$

Hence to find the FWHM, we solve for the values of ω at which $I(\omega) = \frac{1}{2}I(\omega_0)$, i.e.:

$$\frac{I(\omega_0)\frac{\tau^2\omega_0^4}{4}}{(\omega - \omega_0)^2 + \frac{\tau^2\omega_0^4}{4}} = \frac{1}{2}I(\omega_0)$$

$$\frac{\tau^2\omega_0^4}{4} = \frac{1}{2}\left((\omega - \omega_0)^2 + \frac{\tau^2\omega_0^4}{4}\right)$$

$$\frac{\tau^2\omega_0^4}{2} - \frac{\tau^2\omega_0^4}{4} = (\omega - \omega_0)^2$$

$$\omega - \omega_0 = \pm\sqrt{\frac{\tau^2\omega_0^4}{4}}$$

$$\omega = \omega_0 \pm \frac{\tau\omega_0^2}{2}$$

The FWHM of the intensity peak is therefore $2\left(\dfrac{\tau\omega_0^2}{2}\right) = \tau\omega_0^2$.

7e. The characteristic timescale of the scattering process is on the order of $\frac{2}{\tau\omega_0^2}$. Let $\Delta t = \frac{2}{\tau\omega_0^2}$ and $\Delta E = \hbar\Delta\omega$ where $\Delta\omega$ is the FWHM found in part (d). What is $\Delta E\Delta t$, and what significance does this value have? (2 marks)

Based on the result of part (d), $\Delta E = \hbar\Delta\omega = \hbar\tau\omega_0^2$.

Therefore, $\Delta E\Delta t = \hbar\tau\omega_0^2\left(\dfrac{2}{\tau\omega_0^2}\right) = 2\hbar$.

This satisfies the energy-time uncertainty principle $\Delta E\Delta t \geq \frac{\hbar}{2}$.

Some additional notes on this question:

7a. This equation is an approximation to the differential equation $\ddot{x} + \omega_0^2 x - \tau\dddot{x} = Fe^{i\omega t}$ for the movement of a charge under an oscillating electric field of frequency ω, with the $\tau\dddot{x}$ term arising due to the force of radiation reaction. (ω_0 and τ are positive constants.) It can be shown that for $\omega_0\tau \ll 1$, the homogeneous solution (i.e. $F = 0$) takes the form $x = Ae^{i\lambda t}$ with $\lambda \approx \omega_0 + \varepsilon$ where ε is small. The equation can then be solved for ε to obtain $\varepsilon \approx \frac{i\tau\omega_0^2}{2}$. Such an analysis can be used to describe the natural linewidth of spectral lines. (Eyges, L. (1980). *The Classical Electromagnetic Field.*)

7b. It is possible to solve for ϕ as a function of time instead of a constant. However, this would result in a different answer in part (c).

7e. An alternative way to obtain a similar result would be to note that the frequency spectrum of the process can be obtained by the Fourier transform of $x(t) = Ae^{i\lambda t}$, i.e.

$$\begin{aligned}\chi(\omega) = \int_0^\infty x(t)e^{-i\omega t}\,dt &= \int_0^\infty Ae^{i\left(\omega_0 + \frac{i\tau\omega_0^2}{2} - \omega\right)t}\,dt \\ &= \frac{A}{i(\omega_0 - \omega) - \frac{\tau\omega_0^2}{2}}\left[e^{\left(i(\omega_0-\omega) - \frac{\tau\omega_0^2}{2}\right)t}\right]_0^\infty \\ &= \frac{A}{i(\omega - \omega_0) + \frac{\tau\omega_0^2}{2}}\end{aligned}$$

(We have $\lim\limits_{t\to\infty} e^{\left(i(\omega_0-\omega) - \frac{\tau\omega_0^2}{2}\right)t} = 0$ because $\frac{\tau\omega_0^2}{2}$ is a positive real value.)

The intensity is then proportional to $|\chi(\omega)|^2$, leading to a similar result. This also allows a more rigorous analysis of the $\Delta E\Delta t$ relationship; as the functions $\chi(\omega)$ and $x(t)$ in the frequency and time domains respectively are related by a Fourier transform, and there is hence an uncertainty principle between the two functions.

$\frac{2}{\tau\omega_0^2}$ was taken as the characteristic timescale because the solution $x(t) = Ae^{i\lambda t} = Ae^{i\omega_0 t}e^{-\frac{\tau\omega_0^2}{2}t}$ has an exponential-decay factor with time constant $\frac{2}{\tau\omega_0^2}$.

Answer only <u>one</u> out of questions 8 and 9.

8a. Consider a one-dimensional potential barrier defined by the step function:

$$V(x) = \begin{cases} V_0 & \text{for } x < 0 \\ 0 & \text{for } x > 0 \end{cases} \qquad \text{where } V_0 \text{ is a positive real constant}$$

A particle with energy $E < V_0$, travelling from the right, encounters the step at $x = 0$. Show that the incident and reflected probability currents at that point can be expressed in the forms $j_{incident} = a\,|A|^2$ and $j_{reflected} = b\,|B|^2$ respectively. (6 marks)
Note: The probability current at a point is given by

$$j = \frac{i\hbar}{2m}\left(\Psi\frac{d\Psi^*}{dx} - \Psi^*\frac{d\Psi}{dx}\right)$$

In the region $x < 0$:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \implies \frac{d^2\psi}{dx^2} = \frac{2m(V_0 - E)}{\hbar^2}\psi$$

$$\implies \psi = Ae^{\alpha x} + Be^{-\alpha x} \qquad \text{where } \alpha = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

The term $e^{-\alpha x}$ goes to infinity as $x \to -\infty$, and hence we must have $B = 0$ as the wavefunction would otherwise be non-normalizable.

In the region $x > 0$:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \implies \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

$$\implies \psi = Ce^{ikx} + De^{-ikx} \qquad \text{where } k = \frac{\sqrt{2mE}}{\hbar}$$

Since the potential is finite, we require continuity of ψ and $\frac{d\psi}{dx}$ at $x = 0$:

$A = C + D \quad \text{and} \quad \alpha A = ik(C - D)$

$$\implies C = \frac{A}{2}\left(1 - \frac{i\alpha}{k}\right), D = \frac{A}{2}\left(1 + \frac{i\alpha}{k}\right) \qquad (A \text{ remains arbitrary})$$

We note that since $E < V_0$, α is real. Also, we must have $E > 0$ (because if $E < 0$, then $V(x) - E > 0$ for all x; in which case we can see from question 6(b) that $\frac{d^2\psi}{dx^2}$ and ψ have the same sign everywhere and ψ will therefore be non-normalizable), hence k is also real.

The term Ce^{ikx} represents a wave travelling to the right, while the term De^{-ikx} represents a wave travelling to the left. We therefore have

$$\begin{aligned} j_{incident} &= \frac{i\hbar}{2m}\left(\Psi\frac{d\Psi^*}{dx} - \Psi^*\frac{d\Psi}{dx}\right) \qquad \text{with } \Psi = De^{-ikx}e^{-i\omega t}, \omega = \frac{E}{\hbar} \in \mathbb{R} \\ &= \frac{i\hbar}{2m}\left(\left(De^{-ikx}e^{-i\omega t}\right)\left(ikD^*e^{ikx}e^{i\omega t}\right) - \left(D^*e^{ikx}e^{i\omega t}\right)\left(-ikDe^{-ikx}e^{-i\omega t}\right)\right) \quad \text{since } k, \omega \text{ are real} \\ &= -\frac{k\hbar}{m}|D|^2 \quad \text{(this value is negative because the probability current is flowing to the left)} \end{aligned}$$

$$
\begin{aligned}
j_{reflected} &= \frac{i\hbar}{2m}\left(\Psi\frac{d\Psi^*}{dx} - \Psi^*\frac{d\Psi}{dx}\right) \quad \text{with } \Psi = Ce^{ikx}e^{-i\omega t}, \omega = \frac{E}{\hbar} \in \mathbb{R} \\
&= \frac{i\hbar}{2m}\left(\left(Ce^{ikx}e^{-i\omega t}\right)\left(-ikC^*e^{-ikx}e^{i\omega t}\right) - \left(C^*e^{-ikx}e^{i\omega t}\right)\left(ikCe^{ikx}e^{-i\omega t}\right)\right) \quad \text{since } k, \omega \text{ are real} \\
&= \frac{k\hbar}{m}|C|^2
\end{aligned}
$$

8b. Calculate the reflection and transmission coefficients, $R = \left|\frac{j_{reflected}}{j_{incident}}\right|$ and $T = \left|\frac{j_{transmitted}}{j_{incident}}\right|$ respectively. (4 marks)

$$
\begin{aligned}
R = \left|\frac{j_{reflected}}{j_{incident}}\right| &= \left|\frac{\frac{k\hbar}{m}|C|^2}{-\frac{k\hbar}{m}|D|^2}\right| \\
&= \frac{|C|^2}{|D|^2} \\
&= \frac{\left|1 - \frac{i\alpha}{k}\right|^2 |A|^2}{\left|1 + \frac{i\alpha}{k}\right|^2 |A|^2} \\
&= 1
\end{aligned}
$$

Therefore, $T = 1 - R = 0$.

(To confirm:

$$
\begin{aligned}
j_{transmitted} &= \frac{i\hbar}{2m}\left(\Psi\frac{d\Psi^*}{dx} - \Psi^*\frac{d\Psi}{dx}\right) \quad \text{with } \Psi = Ae^{\alpha x}e^{-i\omega t}, \omega = \frac{E}{\hbar} \in \mathbb{R} \\
&= \frac{i\hbar}{2m}\left(\left(Ae^{\alpha x}e^{-i\omega t}\right)\left(\alpha A^*e^{\alpha x}e^{i\omega t}\right) - \left(A^*e^{\alpha x}e^{i\omega t}\right)\left(\alpha Ae^{\alpha x}e^{-i\omega t}\right)\right) \quad \text{since } \alpha, \omega \text{ are real} \\
&= 0
\end{aligned}
$$

Therefore, $T = \left|\frac{j_{transmitted}}{j_{incident}}\right| = 0$.)

8c. Is there transmission through the barrier? (2 marks)

As the transmission coefficient is zero, there is no overall transmission through the barrier. However, there is an exponentially decaying term for the wavefunction in the $x < 0$ region, indicating some probability of finding the particle within that region (though the probability decreases rapidly the further one goes into the barrier).

9a. Write short notes about the comoving coordinate $\vec{x}$ and the scale factor $R(t)$. (1 mark)

- Comoving coordinates are coordinates that follow the expansion of the universe.
- By definition, the comoving coordinate $\vec{x}$ is constant for a free particle following the expansion of the universe.
- The scale factor $R(t)$ relates the physical coordinates $\vec{r}$ to the comoving coordinates.
- The scale factor is a measure of the "size" and expansion rate of the universe.

9b. Write down the expression involving the real distance $\vec{r}$, the co-moving coordinate $\vec{x}$ and the scale factor $R(t)$. (1 mark)

$\vec{r} = R(t)\vec{x}$

9c. The Friedmann equation can be formulated using Newtonian laws.

i. Show that the Friedmann equation takes the form $\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2}$. (4 marks)

Hint: You may wish to consider the kinetic energy of a particle at the edge of the universe moving with speed v.

Assume an infinite homogeneous isotropic universe of density ρ. Define an arbitrary point as the origin. Consider a particle of mass m a distance r from the origin. We note that since $r = Rx$ and x is fixed, we have $\dot{r} = \dot{R}x$.

The mass of the spherical region contained between the particle and the origin is $\frac{4}{3}\pi\rho r^3$. Therefore, it has gravitational potential energy $-\frac{Gm}{r}\left(\frac{4}{3}\pi\rho r^3\right) = -\frac{4\pi G\rho m r^2}{3}$. As its kinetic energy is $\frac{1}{2}m\dot{r}^2$, it has (constant) total energy:

$$E = -\frac{4\pi G\rho m r^2}{3} + \frac{1}{2}m\dot{r}^2 = \frac{4\pi G\rho m R^2 x^2}{3} + \frac{1}{2}m\dot{R}^2 x^2$$

$$\implies \frac{2E}{mR^2x^2} = -\frac{8\pi G\rho}{3} + \left(\frac{\dot{R}}{R}\right)^2, \text{ dividing throughout by } \frac{mR^2x^2}{2}$$

$$\implies \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2}, \text{ introducing the constant } k = -\frac{2E}{mc^2x^2}$$

ii. Write down a formula for k and state what it represents. (1 mark)

As shown above, $k = -\frac{2E}{mc^2x^2}$. It represents the curvature of the universe, with $k > 0$, $k = 0$ and $k < 0$ (i.e. positive, zero or negative curvature) corresponding to closed, flat and open universes respectively.

9d. The fluid equation in cosmology is $\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{P}{c^2}\right)$. Use the result of part (c) to obtain the acceleration equation, $\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$. (3 marks)

Rearranging the result of part (c), $\rho = \frac{3}{8\pi G}\left(\left(\frac{\dot{R}}{R}\right)^2 + \frac{kc^2}{R^2}\right)$

$$\implies \dot{\rho} = \frac{3}{8\pi G}\left(2\left(\frac{\dot{R}}{R}\right)\left(\frac{\ddot{R}R - \dot{R}^2}{R^2}\right) - \frac{2kc^2\dot{R}}{R^3}\right)$$

Substituting the fluid equation $\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{P}{c^2}\right)$ and cancelling the factor $3\frac{\dot{R}}{R}$ from both sides,

$$-\left(\rho + \frac{P}{c^2}\right) = \frac{1}{4\pi G}\left(\frac{\ddot{R}R - \dot{R}^2}{R^2} - \frac{kc^2}{R^2}\right)$$

$$-4\pi G\left(\rho + \frac{P}{c^2}\right) = \frac{\ddot{R}}{R} - \left(\frac{\dot{R}}{R}\right)^2 - \frac{kc^2}{R^2}$$

Substituting the result of part (c) again,

$$-4\pi G\left(\rho + \frac{P}{c^2}\right) = \frac{\ddot{R}}{R} - \left(\frac{8\pi G\rho}{3} - \frac{kc^2}{R^2}\right) - \frac{kc^2}{R^2}$$

$$\frac{\ddot{R}}{R} = -4\pi G\left(\rho + \frac{P}{c^2}\right) + \frac{8\pi G\rho}{3}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

9e. How do you think pressure will affect the acceleration of the universe? (1 mark)

As seen from the acceleration equation, increased pressure corresponds to a larger value of $\ddot{R}$, i.e. a faster acceleration of the universe.

9f. Using the above results, comment on how the acceleration of the universe is affected by its curvature. (1 mark)

It can be seen from the result of part (d) that the acceleration equation does not depend directly on the curvature k. It does, however, indirectly affect the acceleration through its relation to R and ρ via the Friedmann equation. In general, for an open universe ($k < 0$), the acceleration continues forever; for a flat universe ($k = 0$), the acceleration asymptotically approaches zero; for a closed universe ($k > 0$), the expansion slows and reverses into a Big Crunch.

Solutions provided by: Tan Ying Zhe Ernest