NATIONAL UNIVERSITY OF SINGAPORE

PC2130 Quantum Mechanics I

(Semester II: AY 2016-17)

Time Allowed: 2 Hours

INSTRUCTIONS TO STUDENTS

1. Please write your student number only. Do **not** write your name.
2. This assessment paper contains **4** questions and comprises 5 printed pages.
3. Students are required to answer **all** questions.
4. Students should write the answers for each question on a new page.
5. This is a CLOSED BOOK examination.
6. Students are allowed to bring in **one A4-sized** (both sides) sheet of notes.
7. Non-programmable electronic calculators are allowed.

1) Wavefunctions and operators **[10 marks]**

The wavefunction $|\Psi\rangle$ is given as a superposition of quantum states $|A\rangle$ and $|B\rangle$:

$$|\Psi\rangle = a|A\rangle + b|B\rangle$$

a, b are complex constants.

Calculate the probability of measuring the particle in quantum state $|A\rangle$ for:

a) $a = \frac{i}{2}$ [2 marks]

b) $b = \frac{1}{3} + \frac{i}{\sqrt{2}}$ [2 marks]

Let $\hat{Q} = \sum_i q_n \, |q_n\rangle\langle q_n|$ be the operator of an observable Q. Here $|q_n\rangle$ is the nth eigenket of $\hat{Q}$ and q_n is the nth eigenvalue. $|\Psi\rangle$ is the quantum state of the system.

c) What is the physical interpretation of $|\langle q_n|\Psi\rangle|^2$? [2 marks]

d) Let $q_n(x) = \langle x|q_n\rangle$ be the wavefunction of $|q_n\rangle$ in position representation.

Write down an integral that evaluates $\langle q_n|\Psi\rangle$ in position representation. [2 marks]

e) In practice, many individual measurements are required to establish the expectation value $\langle\Psi|\hat{Q}|\Psi\rangle$.

Devise an experimental procedure for measuring the expectation value.

For example: 1) prepare quantum state, 2) measure observable Q, 3) What next? [2 marks]

2) Probability density and probability current within the infinite square well **[17 marks]**

Consider the wavefunction

$$\Psi(x,t) = \frac{1}{\sqrt{N}}\left(sin\left(\frac{\pi}{a}x\right)e^{-i\omega_1 t} - sin\left(\frac{2\pi}{a}x\right)e^{-i\omega_2 t}\right), 0 \leq x \leq a.$$

a) Determine the normalization constant N. [3 marks]

b) Calculate the density in the left half of the well [3 marks]

$$\rho_{left}(t) = \int_0^{a/2} \Psi(x,t)^*\Psi(x,t)\, dx.$$

c) What is the minimal value of $\rho_{left}(t)$? [1 mark]

d) What is the maximal value of $\rho_{left}(t)$? [1 mark]

e) Produce a graph (x-axis: time, y-axis: $\rho_{left}(t)$) and indicate minimal and maximal values of $\rho_{left}(t)$. [2 marks]

f) Calculate the probability current [4 marks]

$$J\left(\frac{a}{2},t\right) = \frac{i\hbar}{2m}\left(\Psi\left(\frac{a}{2},t\right)\partial_x\Psi\left(\frac{a}{2},t\right)^* - \Psi\left(\frac{a}{2},t\right)^*\partial_x\Psi\left(\frac{a}{2},t\right)\right).$$

$$m = \frac{3}{2}\frac{\pi^2\hbar}{a^2\Delta\omega}, \qquad \Delta\omega = \omega_2 - \omega_1$$

g) Confirm that your results obey the continuity equation [3 marks]

$$\partial_t\rho_{left}(t) + J\left(\frac{a}{2},t\right) = 0.$$

3) 1D-Harmonic oscillator **[16 marks]**

The Hamilton operator of the harmonic oscillator is given by

$$\hat{H} = \frac{1}{2m}(\hat{p}^2 + (m\omega\hat{x})^2).$$

The eigenvalue equation yields

$$\hat{H}|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle.$$

We define a raising operator $\hat{A}^\dagger$ and a lowering operator $\hat{A}$

$$\hat{A} = \frac{m\omega\hat{x} + i\hat{p}}{\sqrt{2m\hbar\omega}}, \qquad \hat{A}^\dagger = \frac{m\omega\hat{x} - i\hat{p}}{\sqrt{2m\hbar\omega}},$$

$$[\hat{A}, \hat{A}^\dagger] = 1.$$

a) Show that $\hat{A}|n\rangle = c\,|n-1\rangle$, c is a constant. [3 marks]

b) Show that $c = \sqrt{n}$ [3 marks]

c) Employing the definition of $\hat{A}$ in position representation and the condition $\hat{A}|0\rangle = 0$, calculate the normalized ground state wave function $\langle x|0\rangle = \Psi_0(x)$. [5 marks]

Hint: $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

d) Consider the harmonic oscillator in an external electric field. We assume that the electric field is constant and therefore the electrostatic potential is a linear function of position.

Then, the Hamilton operator is given by:

$$\hat{H} = \frac{1}{2m}(\hat{p}^2 + (m\omega x)^2) + \beta x,$$

where β is a constant indicating the strength of the electrostatic potential.

Calculate the energy eigenvalues. [5 marks]

Hint: Introduce a new variable $x' = x + const.$

4) Stern Gerlach experiment and Larmor precession for spin =1 **[17 marks]**

We define the total Spin operator

$$\hat{S}^2|s,m\rangle = \hbar^2 s(s+1)|s,m\rangle,$$

and spin operator for the z-component:

$$\hat{S}_z|s,m\rangle = \hbar m|s,m\rangle.$$

We define the raising and lowering operators

$$\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y\,,$$

and

$$\hat{S}_\pm|s,m\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}|s,m\pm 1\rangle.$$

Assuming a non-polarized beam of atoms with spin $s = 1$ is guided through a Stern-Gerlach experiment:

a) Derive the operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ in matrix form. [4 marks]

b) Calculate the eigenvectors for $\hat{S}_x$. [3 marks]

c) Considering a non-polarized beam of spin =1 atoms, how many spin states are detected on the screen? [1 mark]

d) What is the probability to measure $m_z = +1$ when the magnetic gradient is along z-direction? [1 mark]

e) What is the probability to measure $m_z = +1$ in two Stern-Gerlach experiments in series, when the first magnetic gradient is along x-direction and passing only $m_x = 0$, and the second field gradient is in the z-direction? [2 marks]

f) Consider a magnetic field in the z-direction, and the system is in the quantum state

$$|\Psi\rangle = \alpha|1,1\rangle \cdot e^{-\frac{iE_+t}{\hbar}} + \beta|1,-1\rangle \cdot e^{-\frac{iE_-t}{\hbar}},\;\; \alpha^2 + \beta^2 = 1$$

$$E_\pm = \mp\gamma B_z\hbar.$$

The spin is measured in a direction given by the unit vector $\hat{\boldsymbol{n}}$ rotated an angle θ away from the z-direction (you may assume the azimuthal angle $\emptyset = 0$).

Calculate the expectation value $\langle\hat{S}_{\hat{n}}\rangle$. [6 marks]

-- End of paper--

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