# PC2130 QM1 0809 Solution

1. The statistical operator for an atom pointing in direction $\vec{e}$ is $\rho = \dfrac{1+\vec{e}}{2}$

   $\text{Prob}(+n) = \text{tr}\left\{|\uparrow_n\rangle\langle\uparrow_n|\,\rho\right\} = \text{tr}\left\{\dfrac{1}{4}(1+n\cdot\vec{\sigma})(1+e\cdot\vec{\sigma})\right\} = \text{tr}\left\{\dfrac{1}{4}[1+\vec{n}\cdot\vec{\sigma}+\vec{e}\cdot\vec{\sigma}+(\vec{n}\cdot\vec{\sigma})(\vec{e}\cdot\vec{\sigma})]\right\}$

   $= \text{tr}\left\{\dfrac{1}{4}[1+\vec{n}\cdot\vec{\sigma}+\vec{e}\cdot\vec{\sigma}+\vec{n}\cdot\vec{e}+i(\vec{n}\times\vec{e})\cdot\vec{\sigma}]\right\} = \dfrac{1}{2}\,(1+\vec{n}\cdot\vec{e})$

   Similarly, $\text{Prob}(-n) = \text{tr}\left\{|\downarrow_n\rangle\langle\downarrow_n|\,\rho\right\} = \text{tr}\left\{\dfrac{1}{4}(1-\vec{n}\cdot\vec{\sigma})(1-\vec{e}\cdot\vec{\sigma})\right\} = \dfrac{1}{2}\,(1-\vec{n}\cdot\vec{e})$

2. (a) If $AA^\dagger = A^\dagger A = 0$ and $A|a_i\rangle = |a_i\rangle a_i$, then

$$(A - a_i I)|a_i\rangle = 0$$

$$[\langle a_i|(A^\dagger - a_i^* I)][(A - a_i I)|a_i\rangle] = 0$$

$$\langle a_i|(A^\dagger A - a_i^* A - a_i A^\dagger + a_i^* a_i)|a_i\rangle = 0$$

$$\langle a_i|(AA^\dagger - a_i A^\dagger - a_i^* A + a_i^* a_i)|a_i\rangle = 0$$

$$[\langle a_i|(A - a_i I)][(A^\dagger - a_i^* I)|a_i\rangle] = 0$$

   which means that $\langle a_i|(A - a_i I) = 0$, or $\langle a_i|$ is an eigenbra of A with eigenvalue $a_i$

   (b) Given

$$A|a\rangle = |a\rangle \text{ and } \langle b|A = b\langle b|$$

   Taking inner product of the first equation with $\langle b|$ and second equation with $|a\rangle$:

$$\langle b|A|a\rangle = a\langle b|a\rangle \text{ and } \langle b|A|a\rangle = b\langle b|a\rangle$$

   Subtracting the two, we get $(a-b)\langle b|a\rangle = 0$. Since $b \neq a$, $\langle b|a\rangle$ is 0.

3. (a) In term of power series, since we have $(\vec{n}\cdot\vec{\sigma})^2 = 1$

$$\begin{aligned} e^{(i\theta\vec{n}\cdot\vec{\sigma})} &= 1 + (i\theta\vec{n}\cdot\vec{\sigma}) + \frac{1}{2!}(i\theta\vec{n}\cdot\vec{\sigma})^2 + \frac{1}{3!}(i\theta\vec{n}\cdot\vec{\sigma})^3 + \cdots + \frac{1}{n!}(i\theta\vec{n}\cdot\vec{\sigma})^n \\ &= \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \dots\right)(\vec{n}\cdot\vec{\sigma}) \\ &= \cos\theta + i\sin\theta(\vec{n}\cdot\vec{\sigma}) \end{aligned}$$

   (b) With $\vec{n} = \frac{1}{\sqrt{3}}(\vec{x}+\vec{y}+\vec{z})$ and $\theta = \frac{2\pi}{3}$

$$\begin{aligned} e^{\left(i\frac{\theta}{2}\vec{n}\cdot\vec{\sigma}\right)} &= \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}(\vec{n}\cdot\vec{\sigma}) = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\left[\frac{1}{\sqrt{3}}(\vec{\sigma}_x+\vec{\sigma}_y+\vec{\sigma}_z)\right] \\ &= \frac{1}{2}\left[1 + i(\sigma_x+\sigma_y+\sigma_z)\right] = \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ i-1 & 1-i \end{pmatrix} \end{aligned}$$

   Acting on $|\uparrow_z\rangle$

$$\frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ i-1 & 1-i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+i \\ i-1 \end{pmatrix}$$

4. Using $\langle x|X = x\langle x|$ and $\langle x|P = -i\hbar\frac{\partial}{\partial x}\langle x|$

$$\langle x|XP = x\langle x|P = -i\hbar x\frac{\partial}{\partial x}\langle x|$$

$$\langle x|PX = -i\hbar\frac{\partial}{\partial x}\langle x|X = -i\hbar\frac{\partial}{\partial x}\langle x|x = -i\hbar x\frac{\partial}{\partial x}\langle x| - i\hbar\langle x|$$

$$\langle x|[X,P] = \langle x|XP - \langle x|PX = i\hbar\langle x|$$

5. For SHO, the Hamiltonian is

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2X^2$$

$$E = \langle H\rangle = \frac{\langle P^2\rangle}{2m} + \frac{1}{2}m\omega^2\langle X^2\rangle = \frac{\delta P^2}{2m} + \frac{1}{2}m\omega^2\delta X^2 = \frac{1}{2}(\frac{\delta P^2}{m} + \omega^2\delta X^2)$$

$$\geq \sqrt{(\frac{\delta P^2}{m})(\omega^2\delta X^2)} = \sqrt{\delta P^2\delta X^2\omega^2} = \frac{1}{2}\hbar\omega$$

6. Using Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} - \frac{\hbar^2\kappa}{m}\delta(x)\psi(x) = E\psi(x)$$

$$\frac{d^2\psi(x)}{dx^2} + 2\kappa\delta(x)\psi(x) = -\frac{2mE}{\hbar^2}\psi(x)$$

Intergrate over a small region $[-\epsilon, \epsilon]$,

$$\frac{d\psi(\epsilon)}{dx} - \frac{d\psi(-\epsilon)}{dx} + 2\kappa\psi(0) = q^2\int_{-\epsilon}^{\epsilon}\psi(x)dx \approx 2\epsilon q^2\psi(0)$$

Let $\epsilon \to 0$

$$\frac{d\psi(0+)}{dx} - \frac{d\psi(0-)}{dx} + 2\kappa\psi(0) = 0$$

To the left and right of the origin, the potential term vanishes, and since the wavefunctions must vanish at infinity,

$$\psi(x) = \begin{cases} Ae^{qx} & x < 0 \\ Ae^{-qx} & x > 0 \end{cases}$$

The following boundary condition must be satisfied:

$$\psi(0+) = \psi(0-)$$

$$\frac{d\psi(x)}{dx}\Big|_{0+} - \frac{d\psi(x)}{dx}\Big|_{0-} + 2\kappa\psi(0) = 0$$

The first condition requires that $A = B = \psi(0)$. The second condition gives

$$-q\psi(0)e^{-qx}\Big|_{0+} - q\psi(0)e^{qx}\Big|_{0-} + 2\kappa\psi(0) = 0$$

$$-q\psi(0) - q\psi(0) + 2\kappa\psi(0) = 0$$

which give rise to $q = \kappa$. There is only one bound state, with energy

$$E = -\frac{\hbar^2 q^2}{2m} = -\frac{\kappa^2 q^2}{2m}$$

The wavefunction can be obtained by normalization:

$$\int_{-\infty}^{\infty} |\psi(0)|^2 e^{-2\kappa|x|} dx = \int_{-\infty}^{0} |\psi(0)|^2 e^{2\kappa x} dx + \int_{0}^{\infty} |\psi(0)|^2 e^{-2\kappa x} dx$$

$$= \frac{|\psi(0)|^2}{2\kappa} \left\{ [e^2 \kappa x] \Big|_{-\infty}^{0} + [-e^2 \kappa x] \Big|_{0}^{\infty} \right\} = \frac{|\psi(0)|^2}{2\kappa} (2) = 1$$

Hence $\psi(0) = \sqrt{k}$. The bound state's wavefunction is

$$\psi(x) = \sqrt{k} e^{-\kappa|x|}$$