NATIONAL UNIVERSITY OF SINGAPORE

PC2130 Quantum Mechanics 1

(Semester I: AY 2008-09)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **6 short** questions in Part I and **3 long** questions in Part II. It comprises **10** printed pages, including this cover page.
2. Answer **ALL** the questions in Part I. The answers to Part I are to be written on the question paper itself and submitted at the end of the examination.
3. Answer any **TWO** of the questions in Part II. The answers to Part II are to be written in the answer books.
4. This is a **CLOSED BOOK** examination. Students are allowed to bring in an A4-sized (both sides) sheet of notes.
5. The total mark for Part I is 48 and that for Part II is 52.

IMPORTANT

Matriculation No.	Marks

PC2130 Quantum Mechanics 1
AY 2008-09 Semester 1 Examination
Part I: Short Questions

Attempt ALL Questions in this Part.
Eight (8) Marks per Question.

1. Atoms with their spin pointing in the direction of the unit vector $\mathbf{e}$ are sent through Stern-Gerlach apparatus oriented in the direction of the unit vector $\mathbf{n}$. Determine the fraction of atoms that will be deflected upwards and the fraction of atoms that will be deflected downwards.

2. A normal operator is one which commutes with its adjoint: $A^\dagger A = AA^\dagger$.

(a) Prove that if $|a\rangle$ is an eigenket of A with eigenvalue a, then $\langle a|$ is also an eigenbra of A with the same eigenvalue.

(b) Prove that if $|b\rangle$ is another eigenket of A with eigenvalue b, and $b \neq a$, then $\langle a | b \rangle = 0$.

3. The rotation operator $R_{\mathbf{n}}(\theta) = \mathrm{e}^{i\frac{\theta}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}$.

(a) Show that for a general unit vector $\mathbf{n}$: $\mathrm{e}^{i\theta\mathbf{n}\cdot\boldsymbol{\sigma}} = \cos\theta + i\sin\theta\,(\mathbf{n}\cdot\boldsymbol{\sigma})$.

(b) What is the spin state obtained by rotating $|\uparrow_z\rangle$ 120° about the axis $\mathbf{n} = \frac{1}{\sqrt{3}}(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})$?

4. Evaluate the expressions $\langle x| XP$ and $\langle x| PX$, stating clearly the intermediate steps with relevant explanations or remarks, and hence show that $\langle x| [X, P] = i\hbar \langle x|$.

5. It is known that for a certain state of a simple harmonic oscillator, $\langle X \rangle = 0$ and $\langle P \rangle = 0$, and hence $\delta X^2 = \langle X^2 \rangle$, and $\delta P^2 = \langle P^2 \rangle$. Show that if the state is known to attain the minimum uncertainty $\delta X \delta P = \hbar/2$, then the energy of the state must be at least $\frac{1}{2}\hbar\omega$.

Note: It may be useful to know that the arithmetic mean is not smaller than the geometric mean: $\frac{1}{2}(a+b) \geq \sqrt{ab}$.

6. A particle of mass m is trapped in a delta-potential well

$$V(x) = -\frac{\hbar^2 \kappa}{m}\delta(x)$$

where κ is a constant. Determine the energy level(s) and corresponding wavefunction(s) of the bound state(s).

Part II: Long Questions

Attempt any TWO (2) Questions in this Part.
Twenty Six (26) Marks per Question.

1. **(a)** A spin state $|\phi\rangle$ has its spin pointing in the direction

$$\mathbf{n} = \sin\theta\cos\varphi\,\hat{\mathbf{x}} + \sin\theta\sin\varphi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}},$$

where θ and φ are the polar and azimuthal angles. Show that the spin state may be expressed as a linear combination in the z-basis as

$$|\psi\rangle = |\uparrow_z\rangle\cos(\theta/2) + |\downarrow_z\rangle\sin(\theta/2)\,e^{i\varphi}.$$

(b) A general spin state $|\psi\rangle = |\uparrow_z\rangle\,\alpha + |\downarrow_z\rangle\,\beta$. Show that:

$$\langle\psi|(\sigma_x + i\sigma_y)|\psi\rangle = 2\alpha^*\beta, \qquad \langle\psi|\sigma_z|\psi\rangle = \alpha^*\alpha - \beta^*\beta.$$

(c) The results of Stern-Gerlach measurements of a batch of atoms, all of them prepared in the same spin state, are summarised below:

	x-measurements		y-measurements		z-measurements	
	$+x$	$-x$	$+y$	$-y$	$+z$	$-z$
Percentage of atoms	50%	50%	10%	90%	80%	20%

Determine the spin state of the atoms, and the direction $\mathbf{n}$ of the spin.

Note: $\tan^{-1}(1/2) = 26.57°$.

2. Let $|\psi\rangle$ be a general state, $\psi(x,t) = \langle x, t\,|\,\psi\rangle$ the position wave function, X and P the position and momentum operators, $T = P^2/(2m)$ the kinetic energy operator, and $V(X)$ the potential energy operator (where V is a real function).

Show that:

(a) $$\langle\psi\,|\,XP\,|\,\psi\rangle = -i\hbar \int_{-\infty}^{\infty} dx\; x\,\psi^*(x,t)\,\frac{\partial\psi(x,t)}{\partial x}.$$

(b) $$\langle\psi\,|\,T\,|\,\psi\rangle = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dx\;\psi^*(x,t)\,\frac{\partial^2\psi(x,t)}{\partial x^2}$$
$$= \frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dx\;\frac{\partial\psi^*(x,t)}{\partial x}\,\frac{\partial\psi(x,t)}{\partial x}.$$

(c) $$\langle\psi\,|\,X\tfrac{dV}{dX}\,|\,\psi\rangle = \int_{-\infty}^{\infty} dx\; x\,\frac{dV(x)}{dx}\,\psi^*(x,t)\psi(x,t).$$

(d) $$\frac{d}{dt}\langle\psi\,|\,XP\,|\,\psi\rangle = \frac{\hbar^2}{m}\int_{-\infty}^{\infty} dx\;\frac{\partial\psi^*(x,t)}{\partial x}\,\frac{\partial\psi(x,t)}{\partial x}$$
$$-\int_{-\infty}^{\infty} dx\; x\,\frac{dV(x)}{dx}\,\psi^*(x,t)\psi(x,t)$$
$$= 2\langle\psi\,|\,T\,|\,\psi\rangle - \langle\psi\,|\,X\tfrac{dV}{dX}\,|\,\psi\rangle.$$

(e) For the stationary states of the harmonic oscillator,
$$\langle\psi\,|\,T\,|\,\psi\rangle = \langle\psi\,|\,V\,|\,\psi\rangle.$$

Note: *You may assume contributions from $x = \pm\infty$ all vanish, e.g.,*
$$x\psi^*\frac{\partial^2\psi}{\partial^2 x}\bigg|_{-\infty}^{\infty} = 0.$$

3. Let $\hat{x} = \sqrt{\frac{m\omega}{2\hbar}} X$, $\hat{p} = \sqrt{\frac{1}{2\hbar m\omega}} P$, and $a = \hat{x} + i\hat{p}$, $N = a^\dagger a$.

(a) Show that the simple harmonic oscillator Hamiltonian can be written as

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2 = \hbar\omega \left(N + \frac{1}{2} \right).$$

(b) Show that

(i) $[a, a^\dagger] = 1$.

(ii) $[N, a] = -a$.

(iii) $[N, a^\dagger] = a^\dagger$.

(c) If $|\nu\rangle$ is an eigenvector of N with eigenvalue ν, i.e., $N|\nu\rangle = |\nu\rangle \nu$, prove that:

(i) The eigenvalue ν is a real number and $\nu \geq 0$.

(ii) $a^\dagger |\nu\rangle$ and $a|\nu\rangle$ are also eigenvectors of N.

(iii) $a|\nu\rangle = 0$ if and only if $\nu = 0$.

(iv) The vector $a^\dagger |\nu\rangle$ never vanishes, i.e., $a^\dagger |\nu\rangle \neq 0$.

(v) The eigenvalue ν must be zero or a positive integer.

(d) Based on the earlier results, prove that the eigenvectors $|n\rangle$ of N are also the eigenstates of the Hamiltonian of the simple harmonic oscillator, associated with energy levels $E_n = (n + \frac{1}{2})\hbar\omega$.

(e) Obtain the position wave function of the ground state.

(LH)

— End of Paper —