PC2130 QUANTUM MECHANICS
AY2008/2009 Semester 2

Suggested Solution

Question 1

(i) Infinitely many pure states that are different from $|\psi\rangle$.

(ii) Two outcomes. Another outcomes is orthogonal to the $|\psi\rangle$

$$|\psi\rangle = \sqrt{\frac{1}{3}}|+z\rangle - i\sqrt{\frac{2}{3}}|-z\rangle$$

(iii)

$$P_\psi = |\psi\rangle\langle\psi| = \begin{aligned} &\tfrac{2}{3}|+z\rangle\langle+z| - i\tfrac{\sqrt{2}}{3}|+z\rangle\langle-z| \\ &+i\tfrac{\sqrt{2}}{3}|-z\rangle\langle+z| + \tfrac{1}{3}|-z\rangle\langle-z| \end{aligned}$$

(iv)

$$\rho = \frac{2}{3}|+z\rangle\langle+z| + \frac{1}{3}|-z\rangle\langle-z|$$

Expand to matrix

(v)

$$\langle A\rangle_\rho = tr\{A\rho\}, \langle A\rangle_\psi = tr\{AP_\psi\}$$

and compare.

Question 2

(i) $H = \begin{pmatrix} 2E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & -E \end{pmatrix}$

$S_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(ii) $|u_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$|u_{-1}\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

$|u_1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$

(iii) $|\psi(0)\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

(iv) $|\psi(t)\rangle = \frac{1}{2}\begin{pmatrix} e^{\frac{-2iEt}{\hbar}} \\ -\sqrt{2}e^{\frac{-iEt}{\hbar}} \\ e^{\frac{iEt}{\hbar}} \end{pmatrix}$

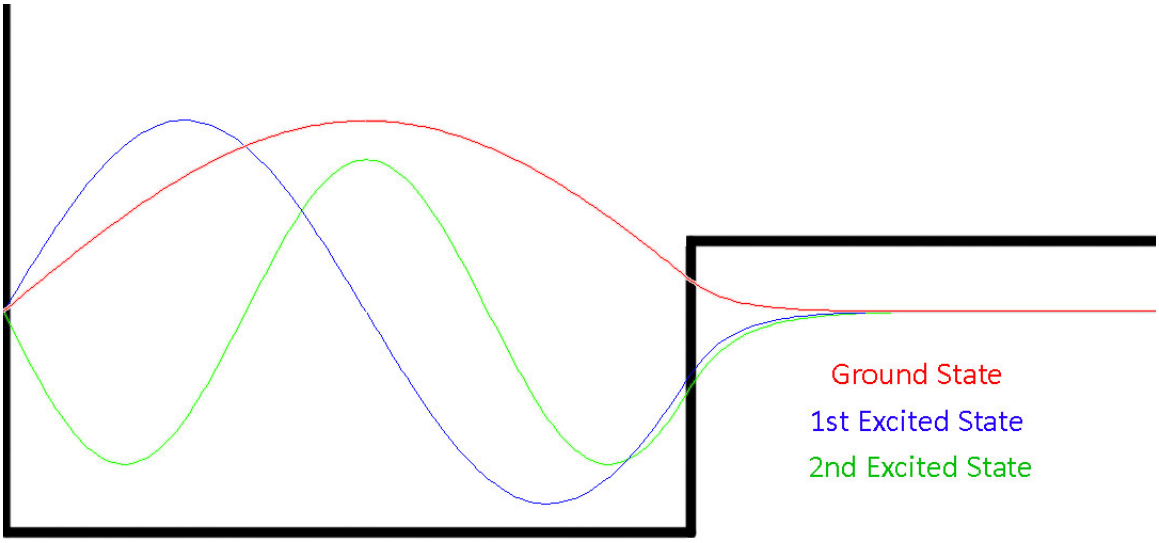
(v) $\langle\psi(t)|S_x|\psi(t)\rangle = -\frac{1}{2}\left(cos\left(\frac{Et}{\hbar}\right) + cos\left(\frac{2Et}{\hbar}\right)\right) = -1$

$\Rightarrow cos\left(\frac{Et}{\hbar}\right) = 1 \quad \text{and} \quad cos\left(\frac{2Et}{\hbar}\right) = 1$

$\Rightarrow t = \frac{2\pi\hbar}{E}$

Question 3

(i) For bound states to exist, we demand $E < 0$.



(ii) Using Schrödinger equation,

$$\begin{cases} \frac{-\hbar^2}{2m}\frac{\partial^2\phi_1}{\partial x^2} - V_0\phi_1 = E\phi_1 & 0 < x < a \\ \frac{-\hbar^2}{2m}\frac{\partial^2\phi_2}{\partial x^2} = E\phi_2 & x > a \end{cases}$$

$$= \begin{cases} \frac{\partial^2\phi_1}{\partial x^2} = \frac{2m(E+V_0)}{-\hbar^2}\phi_1 & 0 < x < a \\ \frac{\partial^2\phi_2}{\partial x^2} = \frac{2mE}{-\hbar^2}\phi_2 & x > a \end{cases}$$

$k_1 = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$

$k_2 = \sqrt{\frac{2mE}{\hbar^2}}$

(iii) Continuity conditions at $x = 0$ and $x = a$,

$$\begin{aligned} \phi_1(0) &= 0 \\ \phi_1(a) &= \phi_2(a) \\ \left.\frac{d\phi_1(x)}{dx}\right|_{x=a} &= \left.\frac{d\phi_2(x)}{dx}\right|_{x=a} \end{aligned}$$

(iv)

$$-i\frac{k_1}{k_2}cot(k_1 a) = \frac{A_2 e^{2ika} - B_2}{A_2 e^{2ika} + B_2}$$

$$\frac{A_2}{B_2} = e^{-2ika}\,\frac{\frac{k_1 i}{k_2}cot(k_1 a) - 1}{\frac{k_1 i}{k_2}cot(k_1 a) + 1}$$

$$\left|\frac{A_2}{B_2}\right|^2 = \frac{\left[\frac{k_1 i}{k_2}cot(k_1 a)\right]^2 + 1}{\left[\frac{k_1 i}{k_2}cot(k_1 a)\right]^2 + 1} = 1$$

(v) For $E > 0$, particle can be found anywhere. Computing $J_k(x)$, we obtain:

$$\begin{aligned} J_1(x) &= 0 \\ J_2(x) &= 0 \end{aligned}$$

Question 4

(i)

$$\begin{aligned} P &= \int_{-a}^{a} \left|\Psi(x)\right|^2 \\ &= \int_{-a}^{a} \Psi^*(x)\Psi(x) \\ &= \int_{-a}^{a} \frac{1}{L} \\ &= \left[\frac{1}{L}x\right]_{-a}^{a} \\ &= \frac{2a}{L} \end{aligned}$$

(ii) $P = 1$

(iii)

$$\begin{aligned} \langle Q \rangle &= \int_{-L/2}^{L/2} x\left|\Psi(x)\right|^2 \\ &= \int_{-L/2}^{L/2} x\frac{1}{L} \\ &= \left[\frac{1}{2L}x^2\right]_{-a}^{a} \\ &= 0 \end{aligned}$$

(iv) $\Psi(k) = \frac{2}{\sqrt{2\pi L}(k_0 - k)} e^{-ik_0 x_0} sin\left(\frac{(k_0 - k)L}{2}\right)$

(v) $\langle P \rangle = \hbar k_0$