NATIONAL UNIVERSITY OF SINGAPORE

PC2130 QUANTUM MECHANICS I

(Semester II: AY 2008-09)

Time allowed: 2 hours

INSTRUCTION TO CANDIDATES

1. This examination paper comprises 4 (four) printed pages in addition to this one.
2. The examination contains 4 (four) questions, all of which should be answered.
3. Answers to the questions are to be written in the answer books.
4. This is a CLOSED BOOK examination.
5. One cheat sheet (A4 size, both sides) is allowed for this examination.

Question 1

This problem deals with two-level systems. We recall the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

We write as $|\pm\hat{z}\rangle$ the eigenvectors of σ_z for the eigenvalues $+1$ and -1 respectively.

Let $|\varphi\rangle = \sqrt{\frac{2}{3}}|+\hat{z}\rangle + i\sqrt{\frac{1}{3}}|-\hat{z}\rangle$:

(i) Among all the possible pure states of a two-level system, how many are *different* from $|\varphi\rangle$: none, one, finitely many, or infinitely many?

(ii) How many outcomes does an ideal measurement of a two-level system have? Supposing one of the outcomes corresponds to $|\varphi\rangle$, give the state(s) corresponding to the other outcome(s).

(iii) Write down the projector P_φ on the state $|\varphi\rangle$.

(iv) Consider now the mixed state

$$\rho = \begin{cases} |+\hat{z}\rangle & \text{with probability } \frac{2}{3} \\ |-\hat{z}\rangle & \text{with probability } \frac{1}{3} \end{cases} :$$

write down the corresponding density matrix.

(v) Do $|\varphi\rangle$ and ρ represent the same state? Justify your answer. *Hint:* the two states are the same if and only if $\langle A\rangle_\rho = \langle A\rangle_\varphi$ for all physical quantities A. Therefore: if your answer is *yes*, provide a proof that the equality indeed holds for any A; if your answer is *no*, exhibit a physical quantity A such that $\langle A\rangle_\rho \neq \langle A\rangle_\varphi$.

Question 2

Consider a three-level system ("spin 1") whose evolution is dictated by the Hamiltonian

$$H = 2E|1\rangle\langle 1| + E|2\rangle\langle 2| - E|3\rangle\langle 3| .$$

The physical quantity "spin along the direction x" is represented by the hermitian operator

$$S_x = \frac{1}{\sqrt{2}}(|1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2|) .$$

(i) Write down H and S_x as matrices, assuming $|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $|3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(ii) Find a set of normalized eigenvectors of S_x and the corresponding eigenvalues.

(iii) At time $t = 0$, an ideal measurement of S_x has yielded the outcome -1: which state $|\psi(0)\rangle$ has been prepared?

(iv) Having this initial state, give the state $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$ at time t.

(v) We repeat many times the following procedure: preparation of $|\psi(0)\rangle$, evolution of the state for a time t, measurement of S_x. For which times (if any) does one find $\langle\psi(t)|S_x|\psi(t)\rangle = \langle\psi(0)|S_x|\psi(0)\rangle = -1$?

Question 3

Consider the eigenvalue problem for the Schrödinger equation $\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\phi_E(x) = E\phi_E(x)$ for the piecewise-constant potential sketched in Figure 1.

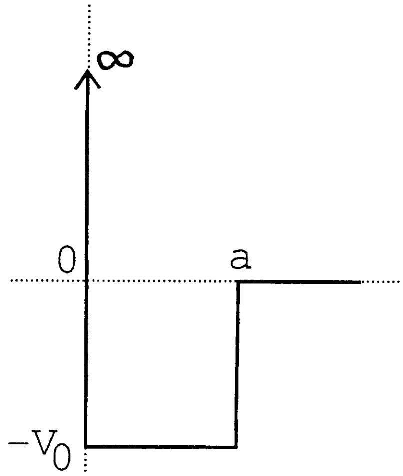

Figure 1: The piecewise-constant potential under study.

(i) Draw qualitatively the form of the stationary states for $-V_0 < E < 0$ and $E > 0$. Discuss in which energy range (if any) the states are bound.

(ii) From now on, consider $E > 0$: the stationary states are of the form

$$\phi_E(x) = \begin{cases} 0 & x < 0 \\ \phi_1(x) = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x} & 0 < x < a \\ \phi_2(x) = A_2 e^{ik_2 x} + B_2 e^{-ik_2 x} & x > a \end{cases} .$$

Find the expressions of $k_1 > 0$ and $k_2 > 0$.

(iii) Write the continuity conditions at $x = 0$ and at $x = a$. *Reminder:* in the presence of an infinite potential step, one of the conditions should not be imposed.

(iv) Compute the reflection coefficient for a wave incoming from the right, $R = \left|\frac{A_2}{B_2}\right|^2$.

(v) For both $\phi_1(x)$ and $\phi_2(x)$, compute the density of probability current

$$J_k(x) = \frac{\hbar}{2mi}\left[\phi_k{}^*(x)\frac{d}{dx}\phi_k(x) - \phi_k(x)\frac{d}{dx}\phi_k{}^*(x)\right] .$$

Question 4

Consider the wave packet

$$\Psi(x) \;=\; \begin{cases} \frac{1}{\sqrt{L}}\, e^{ik_0(x-x_0)} & , \quad x \in [-L/2, L/2] \\ 0 & \quad\text{elsewhere} \end{cases} \;.$$

(i) In a measurement of position, what is the probability of finding the particle in the region $[-a, +a]$ for $a \leq L/2$?

(ii) Same question as in (i) but for $a > L/2$.

(iii) Compute the average value of the position $\langle Q \rangle_\Psi$.

(iv) Compute the Fourier transform of $\Psi(x)$, defined by $\tilde{\Psi}(k) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x) e^{-ikx} dx$.

(v) Compute the average value of the momentum $\langle P \rangle_\Psi$. *Remarks:* if you want to compute in the X basis, neglect the discontinuity of the derivative in $x = \pm L/2$; if you want to compute in the P basis, recall that $p = \hbar k$.

END OF PAPER