PC2130 AY0910 semester 1 final exam solution

Question 1

$$
\begin{aligned}
\mathbf{H} &= \epsilon \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2i \\ 0 & -2i & 1 \end{pmatrix} \\
\mathbf{A} &= \alpha \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}
\end{aligned}
$$

1(a)

Since 3α is the value obtained from measurement A, it is the eigenvalue of matrix A. The eigenvector/eigenstate correspond to this eigenvalue is the initial state of the system:

$$
\begin{aligned}
\begin{pmatrix} \alpha - 3\alpha & \alpha & 0 \\ \alpha & 2\alpha - 3\alpha & 0 \\ 0 & 0 & 3\alpha - 3\alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

Hence, the initial state of the system is, $\quad |\psi(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

The state of the system at later time is then given by:

$$
|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar}\mathbf{H}t\right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

In order to expand $\exp\left(-\frac{i}{\hbar}\mathbf{H}t\right)$, the eigenvalue and eigenvector of $\mathbf{H}$ is needed.

$$
\begin{aligned}
\det |\mathbf{H} - \lambda\mathbf{I}| &= 0 \\
\begin{pmatrix} 2\epsilon - \lambda & 0 & 0 \\ 0 & \epsilon - \lambda & 2i\epsilon \\ 0 & -2i\epsilon & \epsilon - \lambda \end{pmatrix} &= 0 \\
(2\epsilon - \lambda)(\epsilon - \lambda)^2 - 4\epsilon^2(2\epsilon - \lambda) &= 0 \\
(2\epsilon - \lambda)(3\epsilon - \lambda)(-\epsilon - \lambda) &= 0
\end{aligned}
$$

For $\lambda = 2\epsilon$,

$$
\begin{aligned}
\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\epsilon & 2i\epsilon \\ 0 & -2i\epsilon & -\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
|2\epsilon\rangle &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

For $\lambda = -\epsilon$,

$$\begin{pmatrix} 3\epsilon & 0 & 0 \\ 0 & 2\epsilon & 2i\epsilon \\ 0 & -2i\epsilon & 2\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|-\epsilon\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ i \\ -1 \end{pmatrix}$$

For $\lambda = 3\epsilon$,

$$\begin{pmatrix} -\epsilon & 0 & 0 \\ 0 & -2\epsilon & 2i\epsilon \\ 0 & -2i\epsilon & -2\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|3\epsilon\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$$

Hence, the system at later time,

$$\begin{aligned} |\psi(t)\rangle &= \exp\left(-\frac{i}{\hbar}\mathbf{H}t\right)|\psi(0)\rangle \\ |\psi(t)\rangle &= \exp\left(-\frac{i}{\hbar}(-\epsilon)t\right)|-\epsilon\rangle\langle-\epsilon|\,\psi(0)\rangle + \\ &\quad \exp\left(-\frac{i}{\hbar}(2\epsilon)t\right)|2\epsilon\rangle\langle 2\epsilon|\,\psi(0)\rangle + \\ &\quad \exp\left(-\frac{i}{\hbar}(3\epsilon)t\right)|3\epsilon\rangle\langle 3\epsilon|\,\psi(0)\rangle \\ |\psi(t)\rangle &= -\frac{1}{\sqrt{2}}\exp\left(-\frac{i}{\hbar}(-\epsilon)t\right)|-\epsilon\rangle + \frac{1}{\sqrt{2}}\exp\left(-\frac{i}{\hbar}(3\epsilon)t\right)|3\epsilon\rangle \qquad (1) \end{aligned}$$

1(b)

As seen from equation 1, there is only two values will be obtained if we measure **H** at $t > 0$, which are $-\epsilon$ and 3ϵ. The probability of obtaining these values are:

$$\begin{aligned} P(-\epsilon) &= |\langle-\epsilon|\,\psi(t)\rangle|^2 \\ &= \frac{1}{2} \\ P(3\epsilon) &= |\langle 3\epsilon|\,\psi(t)\rangle|^2 \\ &= \frac{1}{2} \end{aligned}$$

1(c)

$$\begin{aligned} \langle H\rangle &= P(-\epsilon)\times(-\epsilon) + P(3\epsilon)\times(3\epsilon) \\ &= \epsilon \end{aligned}$$

The expectation value of the Hamiltonian does not depend on time t, it is because the total energy of the system is conserve.

1(d)

$$\begin{aligned}
\langle A\rangle &= \langle\psi(t)|\,\mathbf{A}\,|\psi(t)\rangle \\
&= \langle\psi(t)|\left(-\frac{1}{\sqrt{2}}\exp\left(-\frac{i}{\hbar}(-\epsilon)t\right)\mathbf{A}\,|-\epsilon\rangle+\frac{1}{\sqrt{2}}\exp\left(-\frac{i}{\hbar}(3\epsilon)t\right)\mathbf{A}\,|3\epsilon\rangle\right) \\
&= \alpha\left[-\frac{1}{\sqrt{2}}\exp\left(-\frac{i}{\hbar}\epsilon t\right)\langle-\epsilon|+\frac{1}{\sqrt{2}}\exp\left(\frac{i}{\hbar}3\epsilon t\right)\langle 3\epsilon|\right] \\
&\quad\times\left[-\frac{1}{2}\exp\left(\frac{i}{\hbar}\epsilon t\right)\begin{pmatrix} i \\ 2i \\ -3\end{pmatrix}+\frac{1}{2}\exp\left(-\frac{i}{\hbar}3\epsilon t\right)\begin{pmatrix} i \\ 2i \\ 3\end{pmatrix}\right] \\
&= \frac{1}{4}\alpha\left[10+2\cos\left(\frac{4\epsilon t}{\hbar}\right)\right]
\end{aligned}$$

The expectation value of observable A depends on time t. It means that it is not a conserve quantity in the system.

Question 2

$$\Psi(x,0) = A\exp\left(-\frac{1}{L}|x|\right)\exp(ik_0x) \qquad A,L,k_0>0$$

2(a)

$$\begin{aligned}
\int_{-\infty}^{\infty}|\Psi(x,0)|^2\,dx &= 1 \\
A^2\int_{-\infty}^{\infty}\exp\left(-\frac{2}{L}|x|\right)dx &= 1 \\
2A^2\int_{0}^{\infty}\exp\left(-\frac{2x}{L}\right)dx &= 1 \\
2A^2\left(-\frac{L}{2}\right)\left[\exp\left(-\frac{2x}{L}\right)\right]_0^{\infty} &= 1 \\
A &= \sqrt{\frac{1}{L}} \\
\Psi(x,0) &= \sqrt{\frac{1}{L}}\exp\left(-\frac{1}{L}|x|\right)\exp(ik_0x)
\end{aligned}$$

2(b)

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty}\Psi^*(x,0)\,x\Psi(x,0)\,dx \\
&= \frac{1}{L}\int_{-\infty}^{\infty}x\exp\left(-\frac{2}{L}|x|\right)dx
\end{aligned}$$

Since x is an odd function, $\exp\left(-\frac{2}{L}|x|\right)$ is an even function, the result of integration above is 0. Hence,

$$\begin{aligned}
\langle x\rangle &= 0 \\
\langle p\rangle &= m\frac{d\langle x\rangle}{dt} \qquad \text{Ehrenfest's Theorem} \\
\langle p\rangle &= 0
\end{aligned}$$

2(c)

$$
\begin{aligned}
\phi(k,0) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,0)\exp(-ikx)\,dx \\
&= \frac{1}{\sqrt{2\pi L}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{L}|x|\right)\exp(i(k_0-k)x)\,dx \\
&= \frac{1}{\sqrt{2\pi L}}\left[\int_{-\infty}^{0}\exp\left(\left(\frac{1}{L}+i(k_0-k)\right)x\right)dx + \right. \\
&\quad \left.\int_{0}^{\infty}\exp\left(\left(-\frac{1}{L}+i(k_0-k)\right)x\right)dx\right] \\
&= \frac{1}{\sqrt{2\pi L}}\left[\frac{1}{\frac{1}{L}+i(k_0-k)}\exp\left(\left(\frac{1}{L}+i(k_0-k)\right)x\right)\Big|_{-\infty}^{0}\right. \\
&\quad \left.+\frac{1}{-\frac{1}{L}+i(k_0-k)}\exp\left(\left(-\frac{1}{L}+i(k_0-k)\right)x\right)\Big|_{0}^{\infty}\right] \\
\phi(k,0) &= \frac{2L}{\sqrt{2\pi L}}\frac{1}{1+L^2(k_0-k)^2}
\end{aligned}
$$

$$
\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k,0)\exp(i(kx-\omega t))\,dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{2L}{\sqrt{2\pi L}}\frac{1}{1+L^2(k_0-k)^2}\exp(ikx)\exp(-i\omega t)\,dk \qquad (2)
\end{aligned}
$$

where $\omega = \frac{\hbar k^2}{2m}$.

2(d)

Since

$$
\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\Phi(k,t)\exp(ikx)\,dk \qquad (3)
$$

By comparing equation 2 and 3, one has:

$$
\Phi(k,t) = \frac{2L}{\sqrt{2\pi L\hbar}}\frac{1}{1+L^2(k_0-k)^2}\exp(-i\omega t)
$$

2(e)

The expectation value of momentum, p is

$$
\begin{aligned}
\langle p\rangle &= \int_{-\infty}^{\infty}p\,|\Phi(k,t)|^2\,dp \\
&= \frac{2L}{\pi\hbar}\int_{-\infty}^{\infty}\frac{\hbar^2 k}{1+L^2(k_0-k)^2}dk \\
&= \frac{2L\hbar}{\pi}\frac{\pi}{2}\frac{k_0}{L} \\
&= p_0
\end{aligned}
$$

The expectation value of position, x is

$$
\begin{aligned}
m\frac{d\langle x\rangle}{dt} &= \langle p\rangle \\
\frac{d\langle x\rangle}{dt} &= \frac{p_0}{m} \\
\langle x\rangle &= \frac{p_0}{m}t + C
\end{aligned}
$$

Since $\langle x\rangle = 0$ when $t = 0$, hence, $\langle x\rangle = \frac{p_0}{m}t$

It means that the particle is moving to the direction of p_0 with a constant velocity.

2(f)

$$
\begin{aligned}
\left\langle p^2\right\rangle &= \int_{-\infty}^{\infty} p^2 \left|\Phi(k,t)\right|^2 dp \\
&= \frac{2L}{\pi\hbar}\int_{-\infty}^{\infty} \frac{\hbar^3 k^2}{1 + L^2\left(k_0 - k\right)^2} dk \\
&= \frac{2L\hbar^2}{\pi}\frac{\pi}{2}\left(\frac{1}{L^3} + \frac{k_0^2}{L}\right) \\
&= \frac{\hbar^2}{L^2} + p_0^2 \\
\sigma_p &= \sqrt{\langle p^2\rangle - \langle p\rangle^2} \\
&= \frac{\hbar}{L}
\end{aligned}
$$

Since $\sigma_x\sigma_p \geq \hbar/2$, Hence, the uncertainty of x is $\sigma_x \geq \frac{L}{2}$. That is, the minimum uncertainty of x is L/2. It is because the "width" of the wave packet is L ($L/2$ for both $+x$ and $-x$ direction). The uncertainty of the position might be higher as time evolve. It is due to the spread of the wave packet itself.

Question 3

$$
\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2
$$

3(a)

$$
\begin{aligned}
\frac{d}{dt}\langle x\rangle &= \frac{i}{\hbar}\langle\Psi(t)|\left[\hat{H},\hat{x}\right]|\Psi(t)\rangle \\
&= \frac{i}{\hbar}\langle\Psi(t)|\left[\frac{1}{2m}\hat{p}^2,\hat{x}\right]|\Psi(t)\rangle + \langle\Psi(t)|\left[\frac{1}{2}m\omega^2\hat{x}^2,\hat{x}\right]|\Psi(t)\rangle \\
&= \frac{i}{2m\hbar}\langle\Psi(t)| - 2i\hbar\hat{p}\,|\Psi(t)\rangle \\
&= \frac{1}{m}\langle p\rangle \\
\frac{d}{dt}\langle p\rangle &= \frac{i}{\hbar}\langle\Psi(t)|\left[\hat{H},\hat{p}\right]|\Psi(t)\rangle \\
&= \frac{i}{\hbar}\langle\Psi(t)|\left[\frac{1}{2m}\hat{p}^2,\hat{p}\right]|\Psi(t)\rangle + \langle\Psi(t)|\left[\frac{1}{2}m\omega^2\hat{x}^2,\hat{p}\right]|\Psi(t)\rangle \\
&= \frac{im\omega^2}{2\hbar}\langle\Psi(t)|\,2i\hbar\hat{x}\,|\Psi(t)\rangle \\
&= -m\omega^2\langle x\rangle
\end{aligned}
$$

3(b)

Using the result from question 3(a),

$$\begin{aligned}\frac{d\langle x\rangle}{dt} &= \frac{1}{m}\langle p\rangle \\ \frac{d^2\langle x\rangle}{dt^2} &= \frac{1}{m}\frac{d\langle p\rangle}{dt} \\ \frac{d^2\langle x\rangle}{dt^2} &= -\omega^2\langle x\rangle\end{aligned}$$

This is the equation of motion of a simple harmonic oscillation system, its solution is given by:

$$\begin{aligned}\langle x\rangle &= C\sin\omega t + D\cos\omega t &(4)\\ \frac{d\langle x\rangle}{dt} &= \omega\left(C\cos\omega t - D\sin\omega t\right) &(5)\\ \frac{1}{m}\langle p\rangle &= \omega\left(C\cos\omega t - D\sin\omega t\right) &(6)\end{aligned}$$

Given that at time $t=0$, $\langle x\rangle = A$ and $\langle p\rangle = 0$, from equation 5 and 6,

$$\begin{aligned}\omega\left(C\cos 0 - D\sin 0\right) &= 0\\ C &= 0\\ D\cos 0 &= A\\ D &= A\end{aligned}$$

$$\text{Hence,}\qquad \langle x\rangle = A\cos\omega t$$

3(c)

$$\begin{aligned}\hat{a} &\equiv \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}\\ \hat{a}^\dagger &\equiv \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}\end{aligned}$$

(i)

$$\begin{aligned}\hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) &= \hbar\omega\left(\frac{m\omega}{2\hbar}\hat{x}^2 + \frac{1}{2m\hbar\omega}\hat{p}^2 + \frac{i}{2\hbar}\left[\hat{x},\hat{p}\right] + \frac{1}{2}\right)\\ &= \hbar\omega\left(\frac{m\omega}{2\hbar}\hat{x}^2 + \frac{1}{2m\hbar\omega}\hat{p}^2 + \frac{i}{2\hbar}i\hbar + \frac{1}{2}\right)\\ &= \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 = \hat{H}\end{aligned}$$

(ii) Since $\left[\hat{a},\hat{a}^\dagger\right] = 1$ and $\hat{H}\left|n\right\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\left|n\right\rangle = \left(n + \frac{1}{2}\right)\hbar\omega$

$$\begin{aligned}\hat{H}\hat{a}\left|n\right\rangle &= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\hat{a}\left|n\right\rangle\\ &= \hbar\omega\left(\left(\hat{a}^\dagger\hat{a}\right)\hat{a} + \frac{1}{2}\hat{a}\right)\left|n\right\rangle\\ &= \hbar\omega\left(\hat{a}\hat{a}^\dagger\hat{a} - \hat{a} + \frac{1}{2}\hat{a}\right)\left|n\right\rangle\\ &= \hbar\omega\hat{a}\left(\hat{a}^\dagger\hat{a} - 1 + \frac{1}{2}\right)\left|n\right\rangle\\ &= \hbar\omega\hat{a}\left((n-1) + \frac{1}{2}\right)\left|n\right\rangle\\ &= \hbar\omega\left((n-1) + \frac{1}{2}\right)\hat{a}\left|n\right\rangle\end{aligned}$$

Similarly,

$$\begin{aligned}\hat{H}\hat{a}^\dagger|n\rangle &= \hbar\omega\left(\hat{a}^\dagger\left(\hat{a}\hat{a}^\dagger\right)+\frac{1}{2}\hat{a}^\dagger\right)|n\rangle \\ &= \hbar\omega\hat{a}^\dagger\left(\hat{a}^\dagger\hat{a}+1+\frac{1}{2}\right)|n\rangle \\ &= \hbar\omega\left((n+1)+\frac{1}{2}\right)\hat{a}^\dagger|n\rangle\end{aligned}$$

3(d)

Given $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, and,

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right)$$

$$\begin{aligned}\langle n|\hat{x}|n'\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left(\langle n|\hat{a}|n'\rangle+\langle n|\hat{a}^\dagger|n'\rangle\right) \\ &= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n'}\langle n|\,n'-1\rangle+\sqrt{n'+1}\langle n|\,n'+1\rangle\right) \\ &= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n'}\delta_{n,n'-1}+\sqrt{n'+1}\delta_{n,n'+1}\right) \\ \langle n|\hat{x}^2|n'\rangle &= \frac{\hbar}{2m\omega}\langle n|\left(\hat{a}+\hat{a}^\dagger\right)^2|n'\rangle \\ &= \frac{\hbar}{2m\omega}\left(\langle n|\hat{a}^2|n'\rangle+\langle n|\hat{a}\hat{a}^\dagger|n'\rangle+\langle n|\hat{a}^\dagger\hat{a}|n'\rangle+\langle n|\hat{a}^{\dagger 2}|n'\rangle\right) \\ &= \frac{\hbar}{2m\omega}\left(\sqrt{n'(n'-1)}\langle n|\,n'-2\rangle+(n'+1)\langle n|\,n'\rangle+(n')\langle n|\,n'\rangle\right. \\ &\quad\left.+\sqrt{(n'+1)(n'+2)}\langle n|\,n'+2\rangle\right) \\ &= \frac{\hbar}{2m\omega}\left(\sqrt{n'(n'-1)}\delta_{n,n'-2}+(2n'+1)\delta_{n,n'}+\sqrt{(n'+1)(n'+2)}\delta n,n'+2\right)\end{aligned}$$

3(e)

$$\begin{aligned}\langle 0|\exp(i\hat{x})|0\rangle &= \langle 0|\left(1+i\hat{x}-\frac{1}{2!}\hat{x}^2-\frac{i}{3!}\hat{x}^3+\dots\right)|0\rangle \\ &= 1+i\langle 0|\hat{x}|0\rangle-\frac{1}{2!}\langle 0|\hat{x}^2|0\rangle+\dots \qquad (7)\end{aligned}$$

As seen from the result in question 3(d), when $n=n'=0$, $\langle 0|\hat{x}|0\rangle$ always equal to 0. In fact, for any positive integer k, $\langle 0|\hat{x}^{2k-1}|0\rangle$ is always equal to 0. Hence, equation 7 can be simplify into

$$\langle 0|\exp(i\hat{x})|0\rangle = 1-\frac{1}{2!}\langle 0|\hat{x}^2|0\rangle+\frac{1}{4!}\langle 0|\hat{x}^4|0\rangle-\dots$$

There are two ways to solve this problem: one is through matrix representation, and another is through wave function. Let us start with the formal first. Recall that the matrix element of $\hat{x}$, x_{mn} is equal to $\langle m|\,\hat{x}\,|n\rangle$. So, from the previous question, the matrix representation of $\hat{x}^2$ is:

$$\hat{x}^2 = \frac{\hbar}{2m\omega} \begin{array}{c} n=0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \overset{\begin{array}{ccccc} n'=0 & 1 & 2 & 3 & \dots \end{array}}{\begin{pmatrix} 1 & 0 & \sqrt{2} & 0 & \dots \\ 0 & 3 & 0 & \sqrt{6} & \dots \\ \sqrt{2} & 0 & 5 & 0 & \dots \\ 0 & \sqrt{6} & 0 & 7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}$$

Subsequently, the matrix representation of $\hat{x}^4$, $\hat{x}^6$, and so on, can be found. Note that we are only interested in the element x_{00}. Hence, only the first row and first column is needed.

$$\begin{aligned} \hat{x}^4 &= \hat{x}^2\hat{x}^2 \\ &= \left(\frac{\hbar}{2m\omega}\right)^2 \begin{array}{c} n=0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} \overset{\begin{array}{cccccc} n'=0 & 1 & 2 & 3 & 4 & \dots \end{array}}{\begin{pmatrix} 3 & 0 & 6\sqrt{2} & 0 & \sqrt{24} & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \dots \\ 6\sqrt{2} & \ddots & \ddots & \ddots & \ddots & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \dots \\ \sqrt{24} & \ddots & \ddots & \ddots & \ddots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}} \\ \hat{x}^6 &= \hat{x}^4\hat{x}^2 \\ &= \left(\frac{\hbar}{2m\omega}\right)^3 \begin{array}{c} n=0 \\ 1 \\ \vdots \end{array} \overset{\begin{array}{ccc} n'=0 & 1 & \dots \end{array}}{\begin{pmatrix} 15 & 0 & \dots \\ 0 & \ddots & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}} \\ \hat{x}^8 &= \hat{x}^4\hat{4}^4 \\ &= \left(\frac{\hbar}{2m\omega}\right)^4 \begin{array}{c} n=0 \\ 1 \\ \vdots \end{array} \overset{\begin{array}{ccc} n'=0 & 1 & \dots \end{array}}{\begin{pmatrix} 105 & 0 & \dots \\ 0 & \ddots & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}} \end{aligned}$$

Let $\xi = \frac{\hbar}{2m\omega}$. Observe that $\langle 0|\,\hat{x}^4\,|0\rangle = \xi^2 \times 1 \times 3$, $\langle 0|\,\hat{x}^6\,|0\rangle = \xi^3 \times 1 \times 3 \times 5$, and $\langle 0|\,\hat{x}^8\,|0\rangle = \xi^4 \times 1 \times 3 \times 5 \times 7$, we conclude that $\langle 0|\,\hat{x}^{2k}\,|0\rangle = \xi^k 1 \times 3 \times \ldots \times (2k-1)$. Hence, equation 7 can be further simplified:

$$\begin{aligned} \langle 0|\exp(i\hat{x})\,|0\rangle &= 1 - \frac{1}{2!}\langle 0|\,\hat{x}^2\,|0\rangle + \frac{1}{4!}\langle 0|\,\hat{x}^4\,|0\rangle - \ldots \\ &= 1 - \frac{1}{2!}\xi + \frac{1\times 3}{4!}\xi^2 - \frac{1\times 3\times 5}{6!} + \ldots \\ &= 1 - \frac{1}{2}\xi + \frac{1}{2\times 4}\xi^2 - \frac{1}{2\times 4\times 6}\xi^3 + \ldots \\ &= 1 - \frac{1}{2\times 1!}\xi + \frac{1}{2^2\times 2!}\xi^2 - \frac{1}{2^3\times 3!}\xi^3 + \ldots \\ &= \sum_{k=0}^{\infty}(-1)^k\frac{1}{2^k k!}\xi^k \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{\xi}{2}\right)^k \\
&= \exp\left(-\frac{\xi}{2}\right) \\
&= \exp\left(-\frac{\hbar}{4m\omega}\right)
\end{aligned}
$$

(Optional) The other way to solve this question 3(e) is through wave function. This requires the Gaussian integral and the wave function of ground state of harmonic oscillator. Since both of them are not given by the question, you might need to remember the formulae. Here, $\xi = \sqrt{\frac{m\omega}{\hbar}}x$

$$
\begin{aligned}
\langle 0|\exp i\hat{x}|0\rangle &= \int_{-\infty}^{\infty} \psi_0^* \exp(ix)\psi_0 dx \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{\infty} \exp(ix)\exp(-\frac{m\omega}{\hbar}x^2 dx) \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left[-\frac{m\omega}{\hbar}\left(x^2 - \frac{i\hbar}{m\omega}x + \left(\frac{i\hbar}{2m\omega}\right)^2 - \left(\frac{i\hbar}{2m\omega}\right)^2\right)\right] dx \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{\hbar}{4m\omega}\right) \int_{-\infty}^{\infty} \exp\left[-\frac{m\omega}{\hbar}\left(x - \frac{i\hbar}{2m\omega}\right)^2\right] dx \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{\hbar}{4m\omega}\right)\left(\sqrt{\pi}\sqrt{\frac{\hbar}{m\omega}}\right) \\
&= \exp\left(-\frac{\hbar}{4m\omega}\right)
\end{aligned}
$$

Question 4

$$
\begin{aligned}
\mathbf{n} &= \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k} \\
|S_n;+\rangle &= \cos\theta\,|S_z;+\rangle + \exp(i\phi)\sin\theta\,|S_z;-\rangle
\end{aligned}
$$

4(a)

Let $|S_n;-\rangle$ spin along the unit vector, $\mathbf{m}$ with polar angle, θ_m and azimuthal angle, ϕ_m. Since $\mathbf{m}$ and $\mathbf{n}$ are parallel but pointing toward opposite directions, one has

$$
\begin{aligned}
\theta_m &= \pi - \theta \\
\phi_m &= \phi + \pi
\end{aligned}
$$

Hence, the states of $|S_n;-\rangle$ is given by:

$$
\begin{aligned}
|S_n;-\rangle &= \cos\frac{\theta_m}{2}|S_z;+\rangle + \exp(i\phi_m)\sin\frac{\theta_m}{2}|S_z;-\rangle \\
&= \cos\left(\frac{-\theta}{2} + \frac{\pi}{2}\right)|S_z;+\rangle \exp(i\pi)\exp(i\phi)\sin\left(\frac{-\theta}{2} + \frac{\pi}{2}\right)|S_z;-\rangle \\
&= \sin\frac{\theta}{2}|S_z;+\rangle - \exp(i\phi)\cos\frac{\theta}{2}|S_z;-\rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle S_n;-| &= \sin\frac{\theta}{2}\langle S_z;+| - \exp(-i\phi)\cos\frac{\theta}{2}\langle S_z;-| \\
\langle S_n;-|\, S_n;+\rangle &= \sin\frac{\theta}{2}\cos\frac{\theta}{2} - \cos\frac{\theta}{2}\sin\frac{\theta}{2} \\
&= 0
\end{aligned}
$$

4(b)

$$
\begin{aligned}
\mathbf{S}_n &= \frac{\hbar}{2}|S_n;+\rangle\langle S_n;+| - \frac{\hbar}{2}|S_n;-\rangle\langle S_n;-| \\
&= \frac{\hbar}{2}\left[\cos^2\frac{\theta}{2}|S_z;+\rangle\langle S_z;+| + \exp(-i\phi)\cos\frac{\theta}{2}\sin\frac{\theta}{2}|S_z;+\rangle\langle S_z;-|\right. \\
&\quad \left. + \exp(i\phi)\cos\frac{\theta}{2}\sin\frac{\theta}{2}|S_z;-\rangle\langle S_z;+| + \sin^2\frac{\theta}{2}|S_z;-\rangle\langle S_z;-|\right] \\
&\quad -\frac{\hbar}{2}\left[\sin^2\frac{\theta}{2}|S_z;+\rangle\langle S_z;+| - \exp(-i\phi)\cos\frac{\theta}{2}\sin\frac{\theta}{2}|S_z;+\rangle\langle S_z;-|\right. \\
&\quad \left. - \exp(i\phi)\cos\frac{\theta}{2}\sin\frac{\theta}{2}|S_z;-\rangle\langle S_z;+| + \cos^2\frac{\theta}{2}|S_z;-\rangle\langle S_z;-|\right] \\
&= \frac{\hbar}{2}[\cos\theta\,|S_z;+\rangle\langle S_z;+| + \exp(-i\phi)\sin\theta\,|S_z;-\rangle\langle S_z;-| \\
&\quad + \exp(i\phi)\sin\theta\,|S_z;-\rangle\langle S_z;+| - \cos\theta\,|S_z;-\rangle\langle S_z;-|\,]
\end{aligned}
$$

4(c)

$$
\begin{aligned}
P\left(\frac{\hbar}{2}\right) &= |\langle S_z;+|\, S_n;+\rangle|^2 \\
&= \cos^2\frac{\theta}{2} \\
P\left(-\frac{\hbar}{2}\right) &= |\langle S_z;-|\, S_n;+\rangle|^2 \\
&= \sin^2\frac{\theta}{2}
\end{aligned}
$$

4(d)

For measurement of $\mathbf{S}_x$,

$$
\begin{aligned}
|S_x;\pm\rangle &= \frac{1}{\sqrt{2}}|S_z;+\rangle \pm \frac{1}{\sqrt{2}}|S_z;-\rangle \\
P\left(\frac{\hbar}{2}\right) &= |\langle S_x;+|\, S_n;+\rangle|^2 \\
&= \frac{1}{2}\left|\cos\frac{\theta}{2} + \exp(i\phi)\sin\frac{\theta}{2}\right|^2 \\
&= \frac{1}{2}(1+\sin\theta\cos\phi) \\
P\left(-\frac{\hbar}{2}\right) &= |\langle S_x;-|\, S_n;+\rangle|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\left|\cos\frac{\theta}{2} - \exp(i\phi)\sin\frac{\theta}{2}\right|^2 \\
&= \frac{1}{2}\left(1 - \sin\theta\cos\phi\right)
\end{aligned}
$$

For measurement of $\mathbf{S}_y$,

$$
\begin{aligned}
|S_y;\pm\rangle &= \frac{1}{\sqrt{2}}\,|S_z;+\rangle \pm \frac{i}{\sqrt{2}}\,|S_z;-\rangle \\
P\left(\frac{\hbar}{2}\right) &= |\langle S_y;+|\,S_n;+\rangle|^2 \\
&= \frac{1}{2}\left|\cos\frac{\theta}{2} + i\exp(i\phi)\sin\frac{\theta}{2}\right|^2 \\
&= \frac{1}{2}\left(1 + \sin\theta\sin\phi\right) \\
P\left(-\frac{\hbar}{2}\right) &= |\langle S_y;-|\,S_n;+\rangle|^2 \\
&= \frac{1}{2}\left|\cos\frac{\theta}{2} - i\exp(i\phi)\sin\frac{\theta}{2}\right|^2 \\
&= \frac{1}{2}\left(1 - \sin\theta\sin\phi\right)
\end{aligned}
$$

4(e)

$$
\begin{aligned}
P(\text{Transmit}) &= |\langle S_n;+|\,S_z;+\rangle|^2 \times |\langle S_x;-|\,S_n;+\rangle|^2 \\
&= \cos^2\frac{\theta}{2} \times \frac{1}{2}\left(1 - \sin\theta\cos\phi\right) \\
&= \frac{1}{2}\cos^2\frac{\theta}{2}\left(1 - \sin\theta\cos\phi\right)
\end{aligned}
$$