NATIONAL UNIVERSITY OF SINGAPORE

PC2130 QUANTUM MECHANICS I

(Semester I: AY 2010-2011)

Time allowed: 2 hours

INSTRUCTION TO CANDIDATES

1. This examination paper comprises 5 (five) printed pages in addition to this one.
2. The examination contains 5 (five) questions, all of which should be answered.
3. Answers to all the questions are to be written in the answer books.
4. This is a CLOSED BOOK examination.
5. One "cheat sheet" (A4 size, single side) is allowed for this examination.
6. This examination will account for 60% of the final mark, and the sum of marks within this examination is 60.

Question 1

Consider a **three-level** system ("spin 1") whose evolution is dictated by the Hamiltonian

$$H = E|1\rangle\langle 1| - E|3\rangle\langle 3| \, .$$

The physical quantity "spin along the direction x" is represented by the Hermitian operator

$$S_x = \frac{\hbar}{\sqrt{2}}(|1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2|) \, ,$$

and the physical quantity "spin along the direction z" is represented by the Hermitian operator

$$S_z = \hbar(|1\rangle\langle 1| - |3\rangle\langle 3|) \, .$$

(a) Write down H, S_x and S_z as matrices, assuming $|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $|3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. [3 marks]

(b) What are the possible outcomes of a measurement of S_x? For each of these measurement outputs, what is the state with which the system ends after the measurement? [3 marks]

(c) What are the possible outcomes of a measurement of S_z? For each of these measurement outputs, what is the state with which the system ends after the measurement? [3 marks]

(d) The physical quantity "spin along the direction y" could be represented by a Hermitian operator S_y, guess what would be the possible outcomes of a measurement of S_y? [1 marks]

(e) At time $t = 0$, a measurement of S_x has yielded the outcome $+\hbar$: which state $|\psi(0)\rangle$ has been prepared? [2 marks]

(f) Having this initial state, give the state $|\psi(t)\rangle$ at time t. [2 marks]

(g) We repeat many times the following procedure: preparation of $|\psi(0)\rangle$, evolution of the state for a time t, measurement of S_x. For which times (if any) does one find $\langle\psi(t)|S_x|\psi(t)\rangle = \langle\psi(0)|S_x|\psi(0)\rangle$? [2 marks]

(h) We repeat many times the following procedure: preparation of $|\psi(0)\rangle$, evolution of the state for a time t, measurement of S_z. For which times (if any) does one find $\langle\psi(t)|S_z|\psi(t)\rangle = \langle\psi(0)|S_z|\psi(0)\rangle$? Explain the physical reason for your answer. [3 marks]

Question 2

Consider a particle of mass m in an infinite potential well:

$$\begin{aligned} V(x) &= 0 \quad \text{if } 0 \leq x \leq a, \\ V(x) &= +\infty \quad \text{if } x < 0 \text{ or } x > a. \end{aligned}$$

$|\phi_n\rangle$ are the eigenstates of the Hamiltonian H of the system, and their eigenvalues are $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$. The state of the particle at time $t = 0$ is

$$|\psi(0)\rangle = a_1|\phi_1\rangle + a_2|\phi_2\rangle + a_3|\phi_3\rangle + a_4|\phi_4\rangle.$$

(a) When the energy of the particle in the state $|\psi(0)\rangle$ is measured, what is the probability of finding the value smaller than $\frac{3\pi^2\hbar^2}{ma^2}$? [2 marks]

(b) What is $\langle E\rangle_\psi$, the mean value of the energy of the particle in the state $|\psi(0)\rangle$? [2 marks]

(c) What is the state $|\psi(t)\rangle$ at time t? [2 marks]

(d) What are the answers to (a) and (b) for the state $|\psi(t)\rangle$ at any arbitrary time t? Explain your results. [4 marks]

(e) When the energy is measured, the result $\frac{8\pi^2\hbar^2}{ma^2}$ is found. After the measurement, what is the state of the system? What is the result if the energy is measured again? [2 marks]

Question 3

For the Hamiltonian of a *harmonic oscillator*: $H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 X^2$, the ground state wave function is $\psi_0(x) = \left(\frac{\beta}{\sqrt{\pi}}\right)^{1/2} e^{-\beta^2 x^2/2}$, where $\frac{1}{\beta} = \sqrt{\frac{\hbar}{m\omega}}$.

(a) Calculate the average value of the position $\langle X \rangle_{\psi_0}$. [2 marks]

(b) Calculate the average value of the momentum $\langle P \rangle_{\psi_0}$. [2 marks]

(c) Calculate the average kinetic energy $\langle E_k \rangle_{\psi_0}$ and the average potential energy $\langle V \rangle_{\psi_0}$, respectively. Comment on your results. [5 marks]

(d) Calculate the uncertainty of the position ΔX_{ψ_0}. [2 marks]

(e) Calculate $\Delta X_{\psi_0} \Delta P_{\psi_0}$. Check if your result is compatible with the uncertainty relation for X and P. [3 marks]

Reminder: $\int_{-\infty}^{+\infty} e^{-y^2} dy = \sqrt{\pi}$ and $\int_{-\infty}^{+\infty} y^2 e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$.

Question 4

A beam of spin-$\frac{1}{2}$ particles with speed v_0 passes through a series of two $\text{SG}_{\mathbf{z}}$ devices. The first $\text{SG}_{\mathbf{z}}$ device transmits particles in the state with the eigenvalue $s_z = +\hbar/2$ and filters out particles in the state with the eigenvalue $s_z = -\hbar/2$. The second $\text{SG}_{\mathbf{z}}$ device transmits particles with $s_z = -\hbar/2$ and filters out particles with $s_z = +\hbar/2$. Between the two devices is a region of length ℓ_0 in which there is a uniform magnetic field B_0 pointing in the x direction. Determine the smallest value of ℓ_0 such that exactly 25 percent of the particles transmitted by the first $\text{SG}_{\mathbf{z}}$ device are transmitted by the second $\text{SG}_{\mathbf{z}}$ device. Express your result in terms of $\omega_0 = egB_0/(2mc)$ and v_0. [10 marks]
Reminder: the Hamiltonian of a spin in magnetic field is $H = -\vec{\mu} \cdot \vec{B}$, where the magnetic dipole moment $\vec{\mu} = \frac{g(-e)}{2mc}\vec{S}$, and $\vec{S} = S_x\hat{x} + S_y\hat{y} + S_z\hat{z}$.

Question 5

Consider a one dimensional system along the x-axis. Find the eigenvalues and the wave functions of the eigenstates (in the x-basis or position space) of the operator $P + \alpha X$ (where α is a real number). *Note: You don't need to calculate explicitly the normalization factor of the wave functions.* [5 marks]

END OF PAPER