NATIONAL UNIVERSITY OF SINGAPORE

PC2130 – Quantum Mechanics I

(Semester I: AY 13-14)

Examiner: Dr. Vitor M. Pereira

Time Allowed: 2 Hours

Instructions to candidates

a) Please write your matriculation/registration number only. **Do not write your name**.

b) This examination paper contains **FOUR (4)** problems and comprises **FOUR (4)** printed pages.

c) This is a **closed book** examination.

d) Answer **ALL** questions.

e) Write your responses in the answer books only. Please start each problem in a new page.

f) Percentage points are indicated beside the header of each problem, together with the partial points for each sub-question.

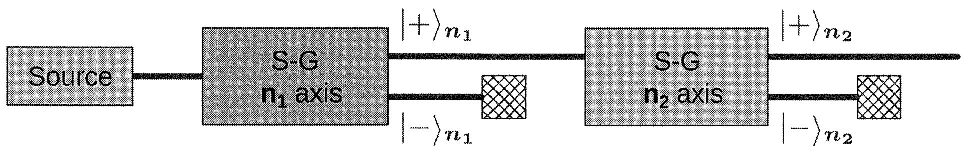
g) No calculators or any other electronic device is allowed during the exam.

h) Students are allowed to bring in an A4-sized (1 side only) sheet of manuscript notes.

i) All answers should be adequately justified.

Problem 1 25 % [5, 5, 5, 10]

Consider a beam of neutral atoms with total spin $S = 1/2$ which is made to go through the sequence of Stern-Gerlach (SG) selectors depicted in the figure below. SG-$\boldsymbol{n}$ represents a Stern-Gerlach device oriented with its field gradient parallel to the direction of the unit vector $\boldsymbol{n}$ that selects the positive eigenstate, $|+\rangle_{\boldsymbol{n}}$, of $\hat{\boldsymbol{S}} \cdot \boldsymbol{n}$, where $\hat{\boldsymbol{S}}$ is the spin operator.



a) Expressing a unit vector $\boldsymbol{n}$ in terms of its spherical angles as

$$\boldsymbol{n} = \sin\theta\cos\varphi\, \boldsymbol{u}_x + \sin\theta\sin\varphi\, \boldsymbol{u}_y + \cos\theta\, \boldsymbol{u}_z,$$

write down a normalized state, $|+\rangle_{\boldsymbol{n}}$, expressed in the eigenbasis of $\hat{S}_z$ that represents the state of the atoms when they leave a SG-$\boldsymbol{n}$ selector.

b) For a general state of a quantum system, $|\psi\rangle$, one can define the density operator

$$\hat{\rho}_\psi = |\psi\rangle\langle\psi| \, .$$

Show that the expectation value of any observable $\hat{O}$ in the state $|\psi\rangle$ can be written in terms of $\hat{\rho}_\psi$ as

$$\langle\psi|\, \hat{O}\, |\psi\rangle = \mathrm{Tr}\left(\hat{\rho}_\psi\, \hat{O}\right).$$

c) Show that the density operator, $\hat{\rho}_{\boldsymbol{n}} = |+\rangle_{\boldsymbol{n}\,\boldsymbol{n}}\langle+|$, associated with the state of the atoms upon leaving a selector SG-$\boldsymbol{n}$ can be written as

$$\hat{\rho}_{\boldsymbol{n}} = \frac{\mathbf{1}}{2} + \frac{1}{\hbar}\hat{\boldsymbol{S}} \cdot \boldsymbol{n}.$$

Suggestion: consider the action of $\hat{\rho}_{\boldsymbol{n}}$ on an arbitrary state expanded in the eigenbasis of $\hat{\boldsymbol{S}} \cdot \boldsymbol{n}$.

d) Consider now the case depicted in the figure above. Using the results above, or otherwise, show that, of all the atoms entering the second selector oriented along $\boldsymbol{n}_2$, the fraction p leaving it can be written as

$$p = \frac{1 + \boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{2},$$

where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are unit vectors defining the orientation of the two selectors.

Hint: it might be useful to recall that Pauli matrices are traceless and, for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the following identity involving the vector of Pauli matrices, $\boldsymbol{\sigma}$, holds: $(\boldsymbol{\sigma} \cdot \boldsymbol{a})\,(\boldsymbol{\sigma} \cdot \boldsymbol{b}) = (\boldsymbol{a} \cdot \boldsymbol{b})\,\mathbf{1} + i\,(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{\sigma}$.

Problem 2 20 % [20]

The time-evolved state of a quantum mechanical system can be generically written in terms of the evolution operator, $\hat{U}(t, 0)$, as

$$|\psi(t)\rangle = \hat{U}(t, 0)\, |\psi(0)\rangle \,.$$

In the position representation this equation can be written as

$$\psi(x, t) = \int_{-\infty}^{+\infty} K(x, t;\, x', 0)\, \psi(x', 0)\, dx',$$

allowing us to calculate the wavefunction $\psi(x, t)$ from the initial one, $\psi(x, 0)$. Calculate the function $K(x, t;\, x', 0)$ for a free particle in 1D.

Note: the following result might be useful:

$$\int_{-\infty}^{+\infty} e^{-iax^2+ibx} dx = \sqrt{\frac{\pi}{ia}}\, e^{\frac{ib^2}{4a}}, \qquad (a > 0).$$

Problem 3 30 % [15, 15]

Consider a particle of mass m confined by the following infinite potential well:

$$V(x) = \begin{cases} +\infty, & x < -a \\ 0, & -a < x < a \,. \\ +\infty, & x > a \end{cases}$$

Suddenly, this potential well is suppressed and immediately replaced by a new one:

$$V(x) = -aV_0\, \delta(x) \qquad (a, V_0 > 0).$$

If the energy is measured right after this new potential is turned on, what is the probability of obtaining the value $E = -\frac{ma^2V_0^2}{2\hbar^2}$ when:

a) the particle was initially in the ground state of the infinite potential well?

b) the particle was initially in the first excited state of the infinite potential well?

Note: you do not need to explicitly evaluate integrals, and can express your final result in terms of $I_a(\alpha, \beta)$ defined as

$$I_a(\alpha, \beta) = \int_0^a \cos(\alpha x) \exp(-\beta x)\, dx.$$

Problem 4 25 % [5, 5, 5, 5, 2+3]

Consider the Hamiltonian for the 1D harmonic oscillator,

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}^2,$$

and denote its eigenstates by $|\varphi_n\rangle$. The annihilation (or destruction) and number operators are defined as

$$a = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{X} + i\sqrt{\frac{1}{2m\hbar\omega}}\,\hat{P}, \qquad \hat{N} = a^\dagger a.$$

a) Use the definition of a above and its action on the eigenstates $|\varphi_n\rangle$ to derive the normalized ground state wavefunction, $\varphi_0(x) = \langle x|\varphi_0\rangle$.

b) Calculate the product of uncertainties $\delta X_n \delta P_n$ when the system is in one of its stationary states $|\varphi_n\rangle$, expressing it as a function of the quantum number n.

c) In view of Heisenberg's uncertainty principle, explain why the *exact* value of the product $\delta X_0 \delta P_0$ in the ground state of the harmonic oscillator can be anticipated without its explicit calculation.

d) Calculate the expectation value of the position, $\langle \hat{X} \rangle$, as a function of time when the initial state of the system is

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}|\varphi_0\rangle + \frac{1}{\sqrt{2}}|\varphi_1\rangle\,.$$

e) Consider the following identity regarding a unitary transformation (a translation) of the position operator

$$e^{i\hat{P}s/\hbar}\hat{X}e^{-i\hat{P}s/\hbar} = \hat{X} + s \qquad (s \in \mathbb{R})\,.$$

i) From this identity, or otherwise, calculate $\left[\hat{X}, e^{i\hat{P}s/\hbar}\right]$ and $\left[a, e^{i\hat{P}s/\hbar}\right]$.

ii) Use the previous commutator to show that

$$\langle\varphi_n|\, e^{i\hat{P}s/\hbar}\,|\varphi_0\rangle = C_n\,\langle\varphi_0|\, e^{i\hat{P}s/\hbar}\,|\varphi_0\rangle\,,$$

and write the value of the constant C_n.

Note: the following result might be useful for this problem:

$$\int_{-\infty}^{+\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}, \qquad (\mathrm{Re}\, a > 0)\,.$$

——— **End of exam paper** ———