NATIONAL UNIVERSITY OF SINGAPORE

PC2131 Electricity and Magnetism I

(Semester II: AY 2015 – 16)

Time Allowed: 2 Hours

INSTRUCTIONS TO STUDENTS

1. Please write your student number only. Do not write your name.
2. This assessment paper contains **4** questions and comprises **4** printed pages.
3. Students are required to answer **ALL** questions. The answers are to be written on the answer books.
4. Students should write the answers for each question on a new page.
5. This is a CLOSED BOOK examination.
6. Programmable calculators are **NOT** allowed.
7. All questions carry equal marks. The total mark is 60.

Question 1

Consider a point dipole P with electric dipole moment $\mathbf{p}$.

(a) Suppose P is located at the origin $\mathcal{O}$. The electric potential at a field point $\mathbf{r} = r\hat{\mathbf{r}}$ due to P is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}.$$

Show that the electric field due to P at $\mathbf{r}$ is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}].$$

[6 marks]

You may find the following results useful:

$$\nabla f(r,\theta,\phi) = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$$

where

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}},$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}.$$

(b) Suppose P is in a nonuniform electric field $\mathbf{E}(\mathbf{r})$. Show that it experiences a force $\mathbf{F}$ given by

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}.$$

[3 marks]

(c) Suppose P is oriented normal to and at distance d from an infinite conducting plane that is grounded. Calculate the force F exerted by the plane on P. Justify your answer.

[6 marks]

Question 2

Consider a sample of linear dielectric material, of electric susceptibility χ_e, subject to an electric field.

(a) The electric potential V at a field point $\mathbf{r}$ due to polarization $\mathbf{P}(\mathbf{r}')$ of the sample is given by

$$V(\mathbf{r}) = \int_{\mathcal{V}} \frac{1}{4\pi\epsilon_0}\frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,d\tau'.$$

i. Show that

$$V(\mathbf{r}) = \int_{\mathcal{V}} \frac{1}{4\pi\epsilon_0}\frac{\rho_b(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau' + \oint_{\mathcal{S}} \frac{1}{4\pi\epsilon_0}\frac{\sigma_b(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,da',$$

where $\mathcal{V}$ is the volume of the sample and $\mathcal{S}$ is the boundary of $\mathcal{V}$.

ii. Find $\rho_b(\mathbf{r}')$ and $\sigma_b(\mathbf{r}')$. Express your answers in terms of $\mathbf{P}(\mathbf{r}')$.

[5 marks]

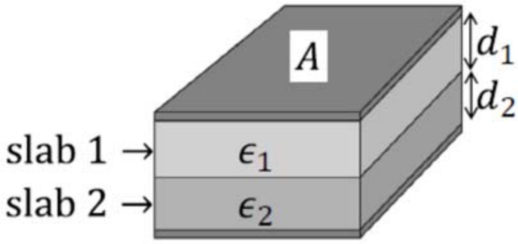
You may find the following results useful:

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = \nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right),$$
$$\nabla\cdot(f\mathbf{A}) = \nabla f\cdot\mathbf{A} + f\nabla\cdot\mathbf{A}.$$

(b) Show that ρ_b is directly proportional to ρ_f, the free charge per unit volume in $\mathcal{V}$. Find the proportionality constant, in terms of χ_e.

[4 marks]

(c) The space between the plates of a parallel-plate capacitor is filled with two slabs of linear dielectric material. Slab 1 has electric permittivity ϵ_1 and thickness d_1, while slab 2 has electric permittivity ϵ_2 and thickness d_2.



Determine the capacitance of the parallel-plate capacitor. Express your answer in terms of area A, d_1, d_2, ϵ_1, and ϵ_2. Justify your answer.

[6 marks]

Question 3

Consider a localized distribution of steady currents described by current density $\mathbf{J}(\mathbf{r}')$. The magnetic field $\mathbf{B}$ at the field point $\mathbf{r}$ due to this current distribution is given by

$$\mathbf{B}(\mathbf{r}) = \int \frac{\mu_0}{4\pi} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\tau'.$$

(a) Show that $\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$.

[6 marks]

You may find the following results useful:

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + (\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B},$$

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot \nabla f,$$

$$\nabla \cdot \left(\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) = 4\pi\delta^3(\mathbf{r} - \mathbf{r}').$$

(b) Show that $\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$.

[3 marks]

You may find the following result useful:

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}).$$

(c) Now consider two long, straight, parallel wires 1 and 2 carrying the same current I, in opposite directions. Show that the associated magnetic vector potential $\mathbf{A}$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{2\pi} \ln \frac{s_2}{s_1} \hat{\mathbf{z}},$$

where s_1 and s_2 are the perpendicular distances from a field point $\mathbf{r}$ to the wires and $\hat{\mathbf{z}}$ is a unit vector parallel to the wires. The current in wire 1 is in the $\hat{\mathbf{z}}$ direction.

[6 marks]

Question 4

(a) Consider a localized distribution of steady currents described by current density $\mathbf{J}(\mathbf{r}')$. The magnetic vector potential $\mathbf{A}$ at the field point $\mathbf{r}$ due to this current distribution is given by

$$\mathbf{A}(\mathbf{r}) = \int \frac{\mu_0}{4\pi} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau'.$$

For $r \gg r'$,

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\theta),$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{r}'$, and $P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$. Suppose $\mathbf{J}(\mathbf{r}')d\tau' = Id\mathbf{l}'$, where $d\mathbf{l}'$ is the line element of a closed wire loop $\mathcal{C}$ carrying current I. $d\mathbf{l}'$ points in the direction of I. Show that

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} + O\left(\frac{1}{r^3}\right),$$

where the magnetic dipole moment

$$\mathbf{m} = I \int d\mathbf{a}'.$$

[5 marks]

You may find the following result useful:

$$\oint_{\partial\mathcal{S}} f d\mathbf{l} = \int_{\mathcal{S}} d\mathbf{a} \times \nabla f.$$

(b) Consider a sample of paramagnetic material subject to a magnetic field. Show that the magnetic vector potential $\mathbf{A}$ at a field point $\mathbf{r}$ due to magnetization $\mathbf{M}(\mathbf{r}')$ of the sample is given by

$$\mathbf{A}(\mathbf{r}) = \int_{\mathcal{V}} \frac{\mu_0}{4\pi} \frac{\mathbf{J}_b(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau' + \oint_{\mathcal{S}} \frac{\mu_0}{4\pi} \frac{\mathbf{K}_b(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} da',$$

where $\mathcal{V}$ is the volume of the sample and $\mathcal{S}$ is the boundary of $\mathcal{V}$. Find $\mathbf{J}_b(\mathbf{r}')$ and $\mathbf{K}_b(\mathbf{r}')$. Express your answers in terms of $\mathbf{M}(\mathbf{r}')$.

[5 marks]

You may find the following results useful:

$$\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} - \mathbf{A} \times \nabla f,$$

$$\int_{\mathcal{V}} (\nabla \times \mathbf{A}) d\tau = -\oint_{\partial\mathcal{V}} \mathbf{A} \times d\mathbf{a}.$$

(c) A solenoid of finite length is filled with a linear paramagnetic material of magnetic susceptibility χ_m.

i. Show that the magnetic field inside the solenoid due to current I in the wire, is greater in magnitude than when the paramagnetic material were absent.

ii. Alice believes that the greater magnetic field is due to the bound current density $\mathbf{J}_b$. Do you agree? Justify your answer.

[5 marks]

– End of Paper –

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