NATIONAL UNIVERSITY OF SINGAPORE

PC2131 Electricity and Magnetism I

(Semester I: AY 2016 – 17)

Time Allowed: 2 Hours

INSTRUCTIONS TO STUDENTS

1. Please write your student number only. Do not write your name.
2. This assessment paper contains **4** questions and comprises **4** printed pages.
3. Students are required to answer **ALL** questions. The answers are to be written on the answer books.
4. Students should write the answers for each question on a new page.
5. This is a CLOSED BOOK examination.
6. Programmable calculators are **NOT** allowed.
7. All questions carry equal marks. The total mark is 60.

Question 1

(a) Verify the following identities for vector fields $\mathbf{F}$ and $\mathbf{G}$:

i. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

ii. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$.

[3]

(b) The Maxwell equations:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho, \qquad \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\mathbf{E},$$

together with the Lorentz force law:

$$\mathbf{F} = \int \rho(\mathbf{E} + \mathbf{v} \times \mathbf{B}) d\tau,$$

form the foundation of classical electrodynamics. Show how the Maxwell equations are consistent with

i. local charge conservation. [4]

ii. local energy conservation. [8]

You should briefly explain what these mean in your answers.

Question 2

(a) If you run a steady current I around a loop of wire C, it produces a magnetic field $\mathbf{B}$. The flux of $\mathbf{B}$ through C, $\Phi = LI$, where L is the self-inductance of C. Show that L depends on the geometry of C.

[5]

(b) It takes a certain amount of work to start a current flowing in a circuit. Starting with zero current, show that the total work done W in order to have a final steady current I running in C is directly proportional to I^2. What is the proportionality constant?

[5]

(c) The total work W done in (b) can be regarded as energy stored in the magnetic field $\mathbf{B}$ associated with the current I. Derive the relationship between W and $\mathbf{B}$.

[5]

Question 3

(a) The Poisson equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{1}{\epsilon_0}\rho(\mathbf{r})$$

is a fundamental equation of electrostatics. Here, $V(\mathbf{r})$ is the electric potential in a volume $\mathcal{V}$ of space with boundary surface $\mathcal{S}$, while $\rho(\mathbf{r})$ describes the charge per unit volume in $\mathcal{V}$. Show that if both $V_1(\mathbf{r})$ and $V_2(\mathbf{r})$ are solutions of the Poisson equation and $V_2(\mathbf{r}) = V_1(\mathbf{r})$ on $\mathcal{S}$, then $V_2(\mathbf{r}) = V_1(\mathbf{r})$ for all $\mathbf{r} \in \mathcal{V}$. You may find the following result useful:

$$\nabla \cdot (f\mathbf{G}) = \nabla f \cdot \mathbf{G} + f\nabla \cdot \mathbf{G}.$$

[6]

(b) Consider a square conducting pipe. The length is much larger than the dimensions of the cross section, which is a square of size $2a \times 2a$. The walls are insulated from each other, as shown in Figure 1.

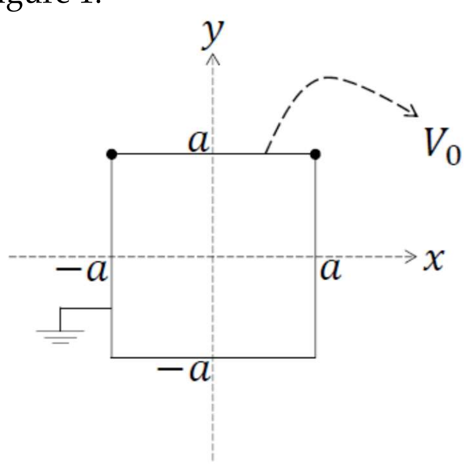

Figure 1

The three sides of the square $x = \pm a$ and $y = -a$ are grounded and the fourth side $y = +a$ is held at a constant potential V_0. Determine the electric potential $V(x, y)$ inside the pipe.

[9]

Question 4

Consider a sphere of dielectric material, centred at the origin O, with radius R. An electric charge Q is placed at $(0, 0, a)$, a point within the sphere, as shown in Figure 2.

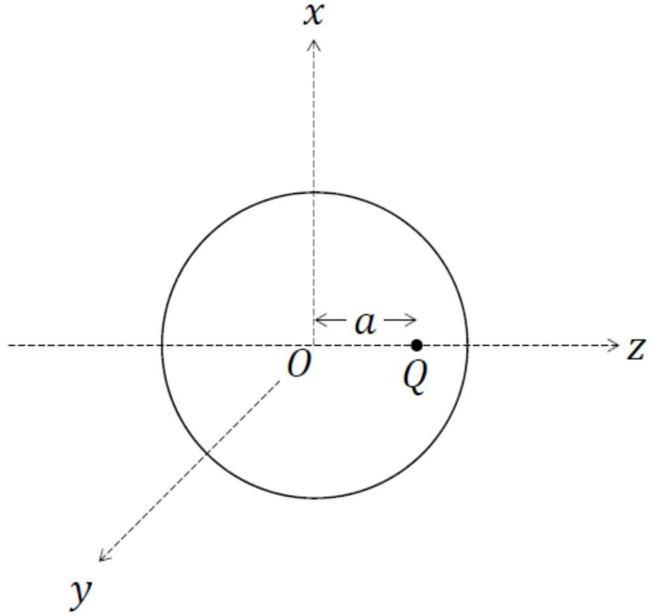

Figure 2

Assume the dielectric is linear and has constant electric susceptibility χ_e. In the following, you may use the result:

$$\nabla \cdot \left(\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) = 4\pi \delta^3(\mathbf{r} - \mathbf{r}'),$$

without proof.

(a) Write down the electric displacement associated with this system. Briefly justify your answer.

[2]

(b) Write down the electric field associated with this system. Briefly justify your answer.

[4]

(c) Calculate the volume charge density ρ_b and surface charge density σ_b associated with the polarisation $\mathbf{P}$ of the dielectric material.

[5]

(d) Calculate the total charge on the surface of the sphere. You may find the following result useful:

$$\int \frac{R - ax}{(R^2 - 2Rax + a^2)^{\frac{3}{2}}} dx = \frac{R\,x - a}{R^2\sqrt{R^2 - 2Rax + a^2}}.$$

[4]

– End of Paper –

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