NATIONAL UNIVERSITY OF SINGAPORE

PC2131 ELECTRICITY & MAGNETISM I

(Semester II: AY 2016-17)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. **Do not write your name.**
2. This examination paper contains **FOUR (4)** questions and comprises **FOUR (4)** printed pages.
3. Students are required to answer **ALL** questions.
4. Students should write the answers for each question on a new page.
5. This is a **CLOSED** book examination. You are allowed however to bring in **ONE** A4-sized help sheet.
6. Specific permitted devices: non-programmable calculators.

Question 1 **[25=10+15]**

The Maxwell equations,

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = \frac{\rho(\mathbf{r},t)}{\epsilon_0}\,, \qquad \nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}\,,$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0\,, \qquad \nabla \times \mathbf{B}(\mathbf{r},t) = \mu_0 \mathbf{J}(\mathbf{r},t) + \mu_0\epsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}\,,$$

together with the Lorentz force law,

$$\mathbf{F} = \iiint_{\mathcal{V}} \rho(\mathbf{r},t)\,[\mathbf{E}(\mathbf{r},\mathbf{t}) + \mathbf{v}(\mathbf{r},t) \times \mathbf{B}(\mathbf{r},t)]\,\mathrm{d}v\,,$$

summarize the entire theoretical content of classical electrodynamics.

(a) Starting from the rate of work done on all charges,

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \iiint_{\mathcal{V}} \mathbf{J}(\mathbf{r},t) \cdot \mathbf{E}(\mathbf{r},t)\,\mathrm{d}v\,,$$

show that the energy density in the fields $u(\mathbf{r},\mathbf{t})$ and the Poynting vector $\mathbf{S}(\mathbf{r},t)$ are given by respectively

$$u(\mathbf{r},t) = \frac{\epsilon_0}{2}\mathbf{E}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t) + \frac{1}{2\mu_0}\mathbf{B}(\mathbf{r},t)\cdot\mathbf{B}(\mathbf{r},t)\,, \qquad \mathbf{S}(\mathbf{r},t) = \frac{1}{\mu_0}\mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)\,.$$

(b) A long cable is made from two coaxial cylindrical shells. The cylindrical axis of the cable is the z-axis. The inner cylindrical conductor, carrying current I_0 in the $+z$ direction, has radius a and charge per unit length λ. The outer cylindrical shell, carrying the return current I_0 in the $-z$ direction, has radius b and charge per unit length $-\lambda$. The currents are uniformly distributed in the conductors.

- **(i)** Find the electric field $\mathbf{E}(\mathbf{r})$ and magnetic field $\mathbf{B}(\mathbf{r})$ between the cylinders.
- **(ii)** Find the Poynting vector $\mathbf{S}(\mathbf{r})$ between the cylinders.
- **(iii)** Find the rate at which energy flows through a cross section of the cable.

Question 2 **[25=10+15]**

(a) An electric dipole $\mathbf{p}_1 = p_0\,\hat{\mathbf{e}}_y$ is located at $\mathbf{r}'_1 = (0, 0, z_0)$ above an infinite, grounded, conducting plane at $z = 0$.

(i) Find the asymptotic electrostatic field $\mathbf{E}(\mathbf{r})$ at an arbitrary point on the z-axis for $z \gg z_0$.

(ii) Find the electrostatic force on the electric dipole $\mathbf{p}_1$.

(b) A conducting sphere of radius a, centered at the origin, is held at a specified electrostatic scalar potential of φ_0. It is surrounded by a concentric spherical shell of radius b with a surface charge density $\sigma(\theta) = \sigma_0 \cos\theta$ where σ_0 is a positive constant.

(i) Find the electrostatic scalar potential $\varphi(\mathbf{r})$ in the regions $a < r < b$ and $r > b$.

(ii) Show that the electrostatic field $\mathbf{E}(\mathbf{r})$ in the region $r < a$ is zero everywhere.

Question 3 **[25=10+15]**

(a) The electrostatic scalar potential of some stationary charge configuration is given by

$$\varphi(\mathbf{r}) = \frac{A}{r}\,\mathrm{e}^{-\lambda r}\,,$$

where A and λ are positive constants.

(i) Find the electrostatic field $\mathbf{E}(\mathbf{r})$ due to this configuration.

(ii) Find the total electric charge Q of this configuration.

(b) A sphere of radius R, centered at the origin, carries uniform volume free charge density ρ. It is filled with a linear isotropic homogeneous dielectric of dielectric constant κ.

(i) Find the electrostatic field $\mathbf{E}(\mathbf{r})$ and electrostatic auxiliary field $\mathbf{D}(\mathbf{r})$ everywhere.

(ii) Find the polarization volume and surface charge distributions.

(iii) Check explicitly that the total polarization volume charge and the total polarization surface charge sum to the expected value.

Question 4 **[25=10+15]**

(a) A localized distribution of steady current is specified by volume current density $\mathbf{J}(\mathbf{r})$. The magnetostatic vector potential $\mathbf{A}(\mathbf{r})$ at a field point $\mathbf{r}$ due to this current distribution is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_{\mathcal{V}'} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}v' \, .$$

The multipole expansion of magnetostatic vector potential for $r \gg r'$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \iiint_{\mathcal{V}'} (r')^n \, P_n(\cos\alpha) \, \mathbf{J}(\mathbf{r}') \, \mathrm{d}v' \, ,$$

where α is the angle between $\mathbf{r}$ and $\mathbf{r}'$, and $P_n(x)$ is the Legendre polynomial. The first few Legendre polynomials are

$$P_0(x) = 1 \, , \qquad P_1(x) = x \, , \qquad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right) \, .$$

A circular loop of wire, of radius a, lies in the xy plane with its center at the origin. It carries a current I running counter-clockwise as viewed from the $+z$ axis.

(i) Find the asymptotic magnetostatic vector potential $\mathbf{A}(\mathbf{r})$ due to the circular loop.

(ii) Hence, or otherwise, find the asymptotic magnetostatic field $\mathbf{B}(\mathbf{r})$ due to the circular loop.

(b) An infinite conducting slab, parallel to the xy plane and extending from $z = -a$ and $z = +a$, carries a uniform free volume current density $\mathbf{J}_\mathrm{f}(\mathbf{r}) = J_0 \, \hat{\mathbf{e}}_x$ where J_0 is a positive constant. The conducting slab is immersed in a large volume of linear isotropic homogeneous magnetic material of magnetic susceptibility χ_m.

(i) Find the magnetostatic field $\mathbf{B}(\mathbf{r})$ and magnetostatic auxiliary field $\mathbf{H}(\mathbf{r})$ everywhere.

(ii) Find the magnetization volume and surface current distributions.

(iii) Check explicitly that the magnetostatic auxiliary field $\mathbf{H}(\mathbf{r})$ satisfies boundary conditions at the surfaces $z = -a$ and $z = +a$ respectively.

KHCM

END OF PAPER