NATIONAL UNIVERSITY OF SINGAPORE

PC2131 Electricity and Magnetism I

(Semester I: AY 2017 – 18)

Time Allowed: 2 Hours

INSTRUCTIONS TO STUDENTS

1. Please write your student number only. Do not write your name.
2. This assessment paper contains **4** questions and comprises **4** printed pages.
3. Students are required to answer **ALL** questions. The answers are to be written on the answer books.
4. Students should write the answers for each question on a new page.
5. This is a CLOSED BOOK examination.
6. Programmable calculators are **NOT** allowed.
7. All questions carry equal marks. The total mark is 60.

Question 1

Consider a sphere of linear dielectric material placed in an otherwise uniform electric field $\mathbf{E}_0$. The sphere has radius R and the electric susceptibility χ_e of the material is a constant.

(a) Find $\mathbf{E}_{\text{in}}$, the electric field inside the sphere.

[5]

(b) Find $\mathbf{E}_{\text{out}}$, the electric field outside the sphere.

[6]

(c) Hence, find $\mathbf{f}$, the force per unit area experienced by charges present on the surface of the sphere.

[4]

Express **ALL FINAL ANSWERS** in forms independent of your choice of coordinates.

You may find the following results useful. In spherical coordinates,

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi},$$

and the Laplace equation for an electric scalar potential $V = V(r,\theta)$ is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0,$$

where $0 \le r < \infty$ and $0 \le \theta \le \pi$. The general solution is given by

$$V(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta).$$

Here, $P_n(\cos\theta)$'s are the Legendre polynomials:

$$P_0(\cos\theta) = 1, \qquad P_1(\cos\theta) = \cos\theta, \qquad P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1), \qquad \cdots.$$

Question 2

Consider a localised distribution of steady current described by volume current density $\mathbf{J} = \mathbf{J}(\mathbf{r})$. The magnetic field $\mathbf{B}$ associated with $\mathbf{J}$ is given by the fundamental equation of magnetostatics,

$$\mathbf{B}(\mathbf{r}) = \int \frac{\mu_0}{4\pi}\frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}d\tau'.$$

(a) Show that $\nabla \cdot \mathbf{J}(\mathbf{r}) = 0$.

[5]

(b) Show that for any two vector fields, $\mathbf{F}$ and $\mathbf{G}$,

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G}).$$

[2]

(c) Hence, show that $\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$.

[3]

(d) Show that for any two vector fields, $\mathbf{F}$ and $\mathbf{G}$,

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} + (\nabla \cdot \mathbf{G})\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} - (\mathbf{F} \cdot \nabla)\mathbf{G}.$$

[2]

(e) Hence, show that $\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0\mathbf{J}(\mathbf{r})$.

[3]

Question 3

(a) A circular loop of wire lies in the xy plane, with its centre at the origin. It has radius R and carries a current I running counterclockwise as viewed from the positive z- axis. The associated magnetic vector potential at a field point $\mathbf{r}$ is given by

$$\mathbf{A}(\mathbf{r}) = \oint \frac{\mu_0 I}{4\pi} \frac{d\boldsymbol{\ell}'}{|\mathbf{r} - \mathbf{r}'|}.$$

Show that at a field point $\mathbf{r}$, such that $|\mathbf{r}| = r \gg R$,

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu_0}{4\pi r^3} \boldsymbol{\mu} \times \mathbf{r},$$

where $\boldsymbol{\mu}$ is to be determined. You may find the following result useful:

$$\frac{1}{\sqrt{1 - 2hx + x^2}} = \sum_{n=0}^{\infty} x^n P_n(h).$$

[6]

(b) Consider a continuous distribution of point magnetic dipoles in a volume $\mathcal{V}$ bounded by surface $\mathcal{S}$. Suppose the magnetic dipole moment per unit volume is given by $\mathbf{M}(\mathbf{r}')$, show that the magnetic vector potential at a field point $\mathbf{r}$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}') d\tau'}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mu_0}{4\pi} \oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r}') da'}{|\mathbf{r} - \mathbf{r}'|},$$

where $\mathbf{J}_b$ and $\mathbf{K}_b$ are to be determined. You may find the following results useful:

$$\nabla \times (f\mathbf{G}) = \nabla f \times \mathbf{G} + f \nabla \times \mathbf{G}, \qquad \int (\nabla \times \mathbf{G}) d\tau = -\oint (\mathbf{G} \times \mathbf{n}) da.$$

[5]

(c) A volume $\mathcal{V}$ is filled with a linear material of constant magnetic susceptibility χ_m. Show that $\mathbf{J}_b = \mathbf{0}$ if there is no free current in $\mathcal{V}$, and the auxiliary field $\mathbf{H} = -\nabla W$ with W satisfying the Laplace equation.

[4]

Question 4

A toroidal coil having a rectangular cross section, an inner radius a, outer radius $a + w$, and height h, has a total of N tightly wound turns.

(a) If it carries a steady current I_0, calculate the associated magnetic field $\mathbf{B}_0$.

[3]

(b) Calculate the total energy E_0 stored in $\mathbf{B}_0$ and hence the self-inductance L of the toroidal coil.

[4]

(c) Now suppose the toroidal coil carries a time-varying current, $I = I(t)$, where t is time. Calculate the associated time-varying electric field $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ in the quasi-static approximation.

[5]

(d) Hence, or otherwise, calculate the electromotive force $\mathcal{E}$ induced in the toroidal coil.

[3]

You may find the following results useful. In cylindrical coordinates,

$$d\boldsymbol{\ell} = dr\hat{\mathbf{r}} + rd\phi\hat{\boldsymbol{\phi}} + dz\hat{\mathbf{z}},$$

where $x = r\cos\phi$, $y = r\sin\phi$, and

$$\hat{\mathbf{r}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}, \qquad \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}.$$

– End of Paper –

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