Question 1(a)

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

Question 1(b)

$$\begin{aligned}\vec{\nabla}\cdot\left(\vec{F}\times\vec{G}\right) &= \frac{\partial}{\partial x_i}\epsilon_{ijk}\left(F_jG_k\right)\\ &= \epsilon_{ijk}\left(\frac{\partial F_j}{\partial x_i}G_k + F_j\frac{\partial G_k}{\partial x_i}\right)\\ &= \epsilon_{kij}\frac{\partial F_j}{\partial x_i}G_k - \epsilon_{jik}F_j\frac{\partial G_k}{\partial x_i}\\ &= \left(\vec{\nabla}\times\vec{F}\right)\cdot\vec{G} - \vec{F}\cdot\left(\vec{\nabla}\times\vec{G}\right)\end{aligned}$$

Question1(c)

Using

$$u = \frac{1}{2}\left(\epsilon_0E^2 + \frac{1}{\mu_0}B^2\right), \qquad \vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)$$

$$\frac{1}{\mu_0}\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) = \frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{E}\right)\cdot\vec{B} - \frac{1}{\mu_0}\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right) = \frac{1}{\mu_0}\left(-\frac{\partial\vec{B}}{\partial t}\right)\cdot\vec{B} - \frac{1}{\mu_0}\vec{E}\cdot\left(\mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$
$$\vec{\nabla}\cdot\vec{S} = -\vec{E}\cdot\vec{J} - \left(\frac{1}{\mu_0}\frac{\partial\vec{B}}{\partial t}\cdot\vec{B} + \epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right)$$

$$\vec{\nabla}\cdot\vec{S} + \frac{\partial u}{\partial t} = -\vec{E}\cdot\vec{J}$$

$$\int\frac{\partial u}{\partial t}dV + \int\vec{\nabla}\cdot\vec{S}\,dV = -\int\vec{E}\cdot\vec{J}\,dV$$
$$\int\frac{\partial u}{\partial t}dV + \oint\vec{S}\cdot d\vec{A} = -\int\vec{E}\cdot\vec{J}\,dV$$

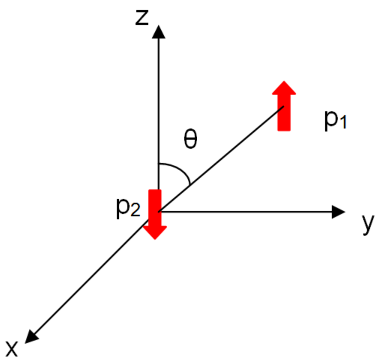
Which is Poynting's theorem. The work done on the charges by the EM force is equal to the decrease in energy stored in the field, less the energy that flowed through the surface.

Question 2(a)
Assuming that the dipole is pointing upwards.

$$\vec{p} = p\hat{z}$$

$$\begin{aligned}
\vec{E} &= -\vec{\nabla}V \\
&= -\vec{\nabla}\left(\frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}\right) \\
&= -\vec{\nabla}\left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2}\right) \\
&= \frac{p\cos\theta}{2\pi\epsilon_0 r^3}\hat{r} + \frac{p\sin\theta}{4\pi\epsilon_0 r^3}\hat{\theta} \\
&= \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)
\end{aligned}$$

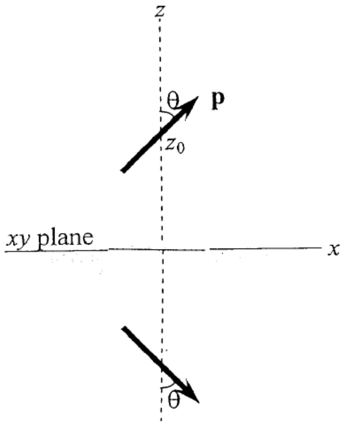
Question 2(b)



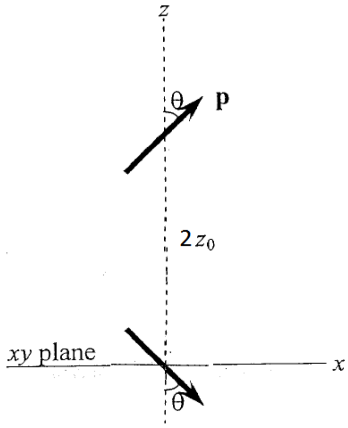
$$\begin{aligned}
\vec{F}_{12} &= (\vec{p}_1\cdot\vec{\nabla})\vec{E}_2 \\
&= \left(p_1\frac{\partial}{\partial r} + \frac{p_1}{r}\frac{\partial}{\partial\theta} + \frac{p_1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\frac{p_2}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \\
&= -\frac{3p_1p_2}{4\pi\epsilon_0 r^4}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) + \frac{p_1p_2}{4\pi\epsilon_0 r^4}\left(-2\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}\right) \\
&= \frac{p_1p_2}{4\pi\epsilon_0 r^4}\left[-2(3\cos\theta + \sin\theta)\hat{r} + (\cos\theta - 3\sin\theta)\hat{\theta}\right]
\end{aligned}$$

Question 2(c)

Using method of images,



We redraw it,



$$\vec{E} = \frac{p}{4\pi\epsilon_0(2z_0)^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$\begin{aligned} N &= \vec{p} \times \vec{E} \\ &= p\left(\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \times \frac{p}{4\pi\epsilon_0(2z_0)^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \\ &= -\frac{p^2 \sin\theta\cos\theta}{32\pi\epsilon_0 z_0^3}\hat{\phi} \end{aligned}$$

Question 3(a)

We know that

$$\nabla_k\left(F_j g\right) = F_j(\nabla_k g) + \left(\nabla_k F_j\right)g$$

Therefore,

$$\begin{aligned} \vec{F} \times \nabla g &= \epsilon_{ijk}F_j(\nabla_k g) \\ &= \epsilon_{ijk}\left[\nabla_k\left(F_j g\right) - \left(\nabla_k F_j\right)g\right] \\ &= -\epsilon_{ikj}\nabla_k\left(F_j g\right) + \epsilon_{ikj}\left(\nabla_k F_j\right)g \\ &= -\vec{\nabla} \times g\vec{F} + \left(\vec{\nabla} \times \vec{F}\right)g \end{aligned}$$

Question 3(b)

We let $\vec{x} - \vec{x}' = \vec{r}$.

$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= \frac{\mu_0}{4\pi}\int \vec{\nabla} \cdot \left(\vec{J} \times \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}\right) dV' \\ &= \frac{\mu_0}{4\pi}\int \vec{\nabla} \cdot \left(\vec{J} \times \frac{\vec{r}}{r^3}\right) dV' \\ &= \frac{\mu_0}{4\pi}\int \frac{\vec{r}}{r^3} \cdot \underbrace{(\vec{\nabla} \times \vec{J})}_{=0} - \vec{J} \cdot \underbrace{\left(\vec{\nabla} \times \frac{\vec{r}}{r^3}\right)}_{=0} dV' \\ &= 0 \end{aligned}$$

$\vec{\nabla} \times \vec{J}$ is zero because $\vec{J}$ does not depend on the unprimed coordinates.

∴ It is solenoidal.

Question 3(c)

$$\vec{\nabla}\cdot\left(g\vec{F}\right)=\nabla_i(gF_i)=(\nabla_i g)F_i+g(\nabla_i F_i)=\vec{\nabla}g\cdot\vec{F}+g\left(\vec{\nabla}\cdot\vec{F}\right)$$

Question 3(d)

$$\begin{aligned}\vec{\nabla}\times\vec{B}(\vec{x})&=\frac{\mu_0}{4\pi}\int\vec{\nabla}\times\left(\vec{J}\times\frac{\vec{r}}{r^3}\right)dV'\\&=\frac{\mu_0}{4\pi}\int\underbrace{\left(\frac{\vec{r}}{r^3}\cdot\vec{\nabla}\right)\vec{J}}_{=0}-\underbrace{\left(\vec{J}\cdot\vec{\nabla}\right)\frac{\vec{r}}{r^3}}_{=0}+\vec{J}\left(\vec{\nabla}\cdot\frac{\vec{r}}{r^3}\right)-\frac{\vec{r}}{r^3}\underbrace{\left(\vec{\nabla}\cdot\vec{J}\right)}_{=0}dV'\\&=\frac{\mu_0}{4\pi}\int\vec{J}(4\pi\delta^3(\vec{r}'))\,dV'\\&=\mu_0\vec{J}\end{aligned}$$

Question 4(a)

Let $\vec{v}\to\vec{v}\times\vec{c}$, where c is a constant.
We know that $\vec{\nabla}\cdot(\vec{v}\times\vec{c})=\vec{c}\cdot\left(\vec{\nabla}\times\vec{v}\right)-\vec{v}\cdot\left(\vec{\nabla}\times\vec{c}\right)=\vec{c}\cdot\left(\vec{\nabla}\times\vec{v}\right)$

$$\begin{aligned}\int_V\vec{\nabla}\cdot(\vec{v}\times\vec{c})\,dV&=\int\vec{c}\cdot\left(\vec{\nabla}\times\vec{v}\right)dV\\&=\oint\vec{v}\times\vec{c}\cdot d\vec{A}\\&=\oint\vec{v}\cdot\vec{c}\times d\vec{A}\\&=-\oint\left(\vec{v}\times d\vec{A}\right)\cdot\vec{c}\end{aligned}$$

We let $\vec{c}=\hat{n}$, then we have

$$\vec{c}\cdot\int\vec{\nabla}\times\vec{v}\,dV=-\vec{c}\cdot\oint\vec{v}\times d\vec{A}$$

$$\therefore\int\vec{\nabla}\times\vec{v}\,dV=-\oint\vec{v}\times d\vec{A}$$

Question 4(b)

$$\vec{\nabla}\times\vec{A}=\vec{B}$$

$$\begin{aligned}\vec{A}&=\frac{\mu_0}{4\pi}\int\vec{M}\times\frac{\vec{r}}{r^3}dV'\\&=\frac{\mu_0}{4\pi}\int\vec{M}\times\left(\vec{\nabla}'\frac{1}{r}\right)dV'\\&=\frac{\mu_0}{4\pi}\left[\int\frac{1}{r}\left(\vec{\nabla}'\times\vec{M}\right)dV'\right]+\frac{\mu_0}{4\pi}\left(\int\vec{\nabla}'\times\frac{\vec{M}}{r}dV'\right)\\&=\frac{\mu_0}{4\pi}\left[\int\frac{1}{r}\left(\vec{\nabla}'\times\vec{M}\right)dV'\right]+\frac{\mu_0}{4\pi}\left(\int\frac{\vec{M}}{r}\times d\vec{A}'\right)\\&=\frac{\mu_0}{4\pi}\int\frac{1}{r}\vec{J}_b\,dV'+\frac{\mu_0}{4\pi}\int\frac{1}{r}\vec{K}_b\,dA'\end{aligned}$$

Question 4(c)

$\vec{M} = M_0 \hat{z}$
$\vec{J}_b = \vec{\nabla} \times \vec{M} = 0$
$\vec{K}_b = \vec{M} \times \hat{n} = M_0 \hat{\phi} = M_0(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})$

$$\vec{r} = \begin{pmatrix} a\cos\phi \\ a\sin\phi \\ z \end{pmatrix}, \qquad r^2 = a^2 + z^2, \qquad \hat{r} = \frac{1}{\sqrt{a^2+z^2}}\begin{pmatrix} a\cos\phi \\ a\sin\phi \\ z \end{pmatrix}$$

Where a is the radius of the cylinder.

$$\vec{K}_b \times \hat{r} = \frac{1}{\sqrt{a^2+z^2}}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -M_0\sin\phi & M_0\cos\phi & 0 \\ a\cos\phi & a\sin\phi & z \end{vmatrix}$$
$$= \frac{1}{\sqrt{a^2+z^2}}(-M_0 z\sin\phi\,\hat{x} - M_0 z\cos\phi\,\hat{y} - M_0 a\hat{z})$$

At the axis of symmetry, $z = 0$.

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}_b \times \hat{r}}{r^2}\,dA, \qquad B_x = B_y = 0$$

$$B_z = \frac{\mu_0 M_0}{4\pi}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^{2\pi} \frac{a}{(a^2+z^2)^{\frac{3}{2}}}\,d\phi\,dz$$
$$= \frac{\mu_0 M_0}{4\pi}(2\pi)\left[\frac{z}{\sqrt{a^2+z^2}}\right]_{-\frac{h}{2}}^{\frac{h}{2}}$$
$$= \frac{2\mu_0 M_0 h}{\sqrt{a^2+h^2}}$$

Solutions done by : John Soo