Question 1(a)

$$\vec{\nabla} f(x,y,z) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

$$\vec{\nabla} \times \vec{\nabla} f(x,y,z) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = 0$$

Since $\frac{\partial f}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x}$ and vice versa.

Question 1(b)

We know that

$$\frac{\partial^2 F_k}{\partial x_i \partial x_j} = \frac{\partial^2 F_k}{\partial x_j \partial x_i}$$

$$\therefore \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = \vec{\nabla} \cdot \left(\epsilon_{ijk} \frac{\partial F_k}{\partial x_j} \right) = \epsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial F_k}{\partial x_j} = \epsilon_{ijk} \frac{\partial^2 F_k}{\partial x_i \partial x_j} = -\epsilon_{jik} \frac{\partial^2 F_k}{\partial x_j \partial x_i} = -\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$$

Question 1(c)

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Question 1(d)

$$\vec{E} = -\vec{\nabla} V(\vec{x}, t) - \frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}(\vec{x}, t)$$

Question 1(e)

$$\vec{\nabla}\cdot\vec{E} = -\nabla^2 V - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \frac{\rho}{\epsilon_0}$$

$$\begin{aligned}\vec{\nabla}\times\vec{B} &= \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) \\ &= \underbrace{\vec{\nabla}(\vec{\nabla}\cdot\vec{A})}_{=0} - \nabla^2\vec{A} \\ &= -\nabla^2\vec{A} \\ &= \mu_0\vec{J} - \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \\ &= \mu_0\vec{J} - \mu_0\epsilon_0\frac{\partial}{\partial t}\vec{\nabla}V - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\end{aligned}$$

So we have 2 equations,

$$-\nabla^2 V - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \frac{\rho}{\epsilon_0}, \qquad (1)$$

$$-\nabla^2\vec{A} = \mu_0\vec{J} - \mu_0\epsilon_0\frac{\partial}{\partial t}\vec{\nabla}V - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}, \qquad (2)$$

From (1),

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\epsilon_0}$$

$$\left(-\mu_0\epsilon_0\frac{\partial^2}{\partial t^2} + \nabla^2\right)V + \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\epsilon_0}$$

$$-\mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} + \nabla^2 V + \frac{\partial}{\partial t}\left(\mu_0\epsilon_0\frac{\partial V}{\partial t} + \vec{\nabla}\cdot\vec{A}\right) = -\frac{\rho}{\epsilon_0}$$

$$\therefore \frac{\partial}{\partial t}\left(\mu_0\epsilon_0\frac{\partial V}{\partial t} + \vec{\nabla}\cdot\vec{A}\right) = \frac{\partial}{\partial t}L_1 \quad \Rightarrow \quad L_1 = \mu_0\epsilon_0\frac{\partial V}{\partial t} + \vec{\nabla}\cdot\vec{A}$$

From (2),

$$\mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2 A + \mu_0\epsilon_0\frac{\partial}{\partial t}\vec{\nabla}V = \mu_0\vec{J}$$

$$\left(-\mu_0\epsilon_0\frac{\partial^2}{\partial t^2} + \nabla^2\right)\vec{A} - \vec{\nabla}\left(\mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J}$$

$$\therefore -\vec{\nabla}\left(\mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\vec{\nabla}L_2 \quad \Rightarrow \quad L_2 = \mu_0\epsilon_0\frac{\partial V}{\partial t}$$

Question 2(a)

Biot-Savart's Law,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r})\times\hat{r}}{r^2}dV'$$

Question 2(b)

$$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\left(\vec{J}(\vec{r})\times\frac{\hat{r}}{r^2}\right)dV' = \frac{\mu_0}{4\pi}\int \frac{\hat{r}}{r^2}(\vec{\nabla}\cdot\vec{J}) - \vec{J}\cdot\left(\vec{\nabla}\times\frac{\hat{r}}{r^2}\right)dV' = 0$$

$\therefore \vec{B}$ is solenoidal.

Question 2(c)

$$\begin{aligned}
\vec{\nabla}\times\vec{B} &= \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left(\vec{J}\times\frac{\hat{r}}{r^2}\right)dV' \\
&= \frac{\mu_0}{4\pi}\int \underbrace{\left(\frac{\hat{r}}{r^2}\cdot\vec{\nabla}\right)\vec{J}}_{=0} - (\vec{J}\cdot\vec{\nabla})\frac{\hat{r}}{r^2} + \underbrace{\vec{J}\left(\vec{\nabla}\cdot\frac{\hat{r}}{r^2}\right)}_{=0} - \underbrace{\frac{\hat{r}}{r^2}(\vec{\nabla}\cdot\vec{J})}_{=0}\,dV' \\
&= \frac{\mu_0}{4\pi}\int \vec{J}[4\pi\delta^3(\vec{r})]\,dV' \\
&= \mu_0\vec{J}
\end{aligned}$$

Question 2(d)

Using multipole expansion, since r is very large,

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta'}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}dV' = \frac{\mu_0}{4\pi}\int\sum_{n=0}^{\infty}\frac{\vec{J}}{r^{n+1}}(r')^n P_n(\cos\theta')\,dV'$$

Since the monopole is zero, $n \geq 1$.

$$\begin{aligned}
\vec{A}(\vec{x}) &= \frac{\mu_0}{4\pi r^2}\int \vec{J}r'\cos\theta'\,dV' + \frac{\mu_0}{4\pi r^3}\int \vec{J}r'^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)dV' + \cdots \\
&\approx \frac{\mu_0}{4\pi r^2}\int \vec{J}r'\cos\theta'\,dV' = \frac{\mu_0}{4\pi r^2}\int \vec{J}(\hat{r}\cdot\vec{x}')dV'
\end{aligned}$$

$$\vec{m} = \frac{1}{2}\int \vec{x}'\times\vec{J}(\vec{x}')\,dV'$$

$$\begin{aligned}
\vec{m}\times\hat{r} &= \frac{1}{2}\int[\vec{x}'\times\vec{J}]\times\hat{r}\,dV' \\
&= \frac{1}{2}\int \epsilon_{lim}\epsilon_{ijk}x_j'J_k\hat{r}_m\,dV' \\
&= \frac{1}{2}\int(\delta_{mj}\delta_{lk}-\delta_{mk}\delta_{lj})x_j'J_k\hat{r}_m\,dV' \\
&= \frac{1}{2}\int x_m'J_l\hat{r}_m - x_l'J_m\hat{r}_m\,dV' = \frac{1}{2}\int x_m'J_l - x_l'J_m\,dV'\hat{r}_m \\
&= \int x_m'J_l\,dV'\hat{r}_m \\
&= \int J_l\hat{r}_m x_m'\,dV' = \int \vec{J}(\hat{r}\cdot\vec{x}')dV'
\end{aligned}$$

$$\therefore \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi r^2}\int\vec{J}(\hat{r}\cdot\vec{x}')dV' = \frac{\mu_0}{4\pi r^2}(\vec{m}\times\hat{r})$$

Question 3(a)

We suppose a surface bounded by charge density ρ. Suppose we can write the solution of V as V_1 and V_2. So we have

$$\nabla^2 V_1 = \nabla^2 V_2 = \frac{\rho}{\epsilon_0}$$

We let $V_3 = V_2 - V_1$. Since the Laplacian operator is linear,

$$\nabla^2 V_3 = \nabla^2 (V_2 - V_1) = \frac{\rho}{\epsilon_0} - \frac{\rho}{\epsilon_0} = 0$$

Since $\nabla^2 V_3 = 0$, it is zero along all boundaries, so $V_3 = 0$. $\therefore V_2 = V_1$, i.e., the potential is uniquely specified, once the potential is specified.

Now we can say

$$\nabla^2 V_1 = \vec{\nabla} \cdot \vec{\nabla} V_1 = -\vec{\nabla} \cdot \vec{E}_1 = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{E}_1 = \vec{\nabla} \cdot \vec{E}_2 = \frac{\rho}{\epsilon_0}, \text{both should obey Gauss'Law.}$$

$$\int (\vec{\nabla} \cdot \vec{E}_1)\, dV = \frac{Q_i}{\epsilon_0} = \oint \vec{E}_2 \cdot d\vec{A} = \oint \vec{E}_1 \cdot d\vec{A} \text{ on the bound areas.}$$

We let $\vec{E}_3 = \vec{E}_2 - \vec{E}_1$.

$$\vec{\nabla} \cdot \vec{E}_3 = \vec{\nabla} \cdot (\vec{E}_2 -\backslash \vec{E}_1) = 0$$

$$\therefore \oint \vec{E}_3 \cdot d\vec{A}$$

We try $\vec{\nabla} \cdot (V_3 \vec{E}_3) = V_3 (\vec{\nabla} \cdot \vec{E}_3) + \vec{E}_3 \cdot \vec{\nabla} V_3 = -E_3^2$

$$\int \vec{\nabla} \cdot (V_3 \vec{E}_3)\, dV = \int E_3^2\, dV = \oint V_3\, \vec{E}_3 \cdot d\vec{A} = 0$$

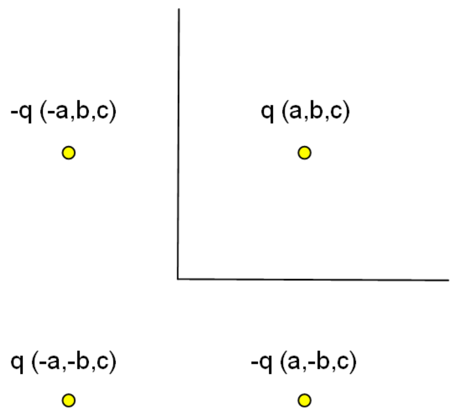
because if V_3 is constant over each surface, $V_3 = 0$ if the outer boundary is infinity.

$$\int E_3^2\, dV = 0, \qquad \vec{E}_2 = \vec{E}_1, \frac{\partial V_2}{\partial n} = \frac{\partial V_1}{\partial n}, \text{uniquely determined.}$$

$\therefore$ Either V or $\dfrac{\partial V}{\partial n}$ is specified on the surface that the potential V is uniquely determined.

Question 3(b)

Using the method of images, we can place another 3 charges of the same magnitude q as follows:



The potential of the system can be written as

$$V(x,y) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}} + \frac{1}{\sqrt{(x+a)^2+(y+b)^2+(z-c)^2}} - \frac{1}{\sqrt{(x-a)^2+(y+b)^2+(z-c)^2}} - \frac{1}{\sqrt{(x+a)^2+(y-b)^2+(z-c)^2}}\right]$$

The charge density on the plane,

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$$

Let

$$\sigma_x = -\epsilon_0 \frac{\partial V}{\partial y}\bigg|_{y=0} = \frac{qb}{2\pi}\left\{\frac{1}{[(x-a)^2+b^2+(z-c)^2]^{\frac{3}{2}}} - \frac{1}{[(x+a)^2+b^2+(z-c)^2]^{\frac{3}{2}}}\right\}$$

$$\sigma_y = -\epsilon_0 \frac{\partial V}{\partial x}\bigg|_{x=0} = \frac{qa}{2\pi}\left\{\frac{1}{[a^2+(y-b)^2+(z-c)^2]^{\frac{3}{2}}} - \frac{1}{[a^2+(y+b)^2+(z-c)^2]^{\frac{3}{2}}}\right\}$$

$$Q_x = \int_0^\infty \int_{-\infty}^\infty \sigma_x \, dz \, dx$$

$$= \int_0^\infty \int_{-\infty}^\infty \frac{qb}{2\pi}\left\{\frac{1}{[(x-a)^2+b^2+(z-c)^2]^{\frac{3}{2}}} - \frac{1}{[(x+a)^2+b^2+(z-c)^2]^{\frac{3}{2}}}\right\} dz \, dx$$

$$= \int_0^\infty \frac{qb}{2\pi}\left[\frac{2}{(x-a)^2+b^2} - \frac{2}{(x+a)^2+b^2}\right] dx$$

$$= \frac{q}{\pi}\left[\frac{\pi}{2} + \tan^{-1}\left(\frac{a}{b}\right) - \frac{\pi}{2} + \tan^{-1}\left(\frac{a}{b}\right)\right]$$

$$= \frac{2q}{\pi}\tan^{-1}\left(\frac{a}{b}\right)$$

Similarly, $Q_y = \frac{2q}{\pi}\tan^{-1}\left(\frac{b}{a}\right)$

∴ total charge, $Q = Q_x + Q_y = \frac{2q}{\pi}\left[\tan^{-1}\left(\frac{a}{b}\right) + \tan^{-1}\left(\frac{b}{a}\right)\right]$

Question 4(a)

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\left(A + \frac{B}{r} + Cr\cos\theta + \frac{D\cos\theta}{r^2}\right)\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(A + \frac{B}{r} + Cr\cos\theta + \frac{D\cos\theta}{r^2}\right)\right]$$

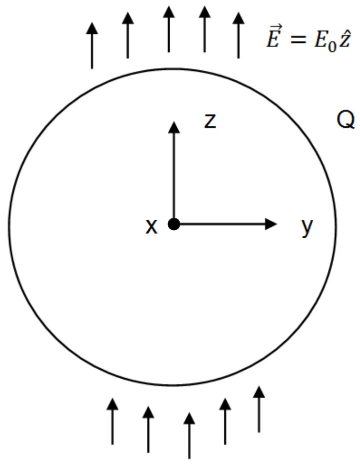
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(-B + Cr^2\cos\theta - \frac{2D\cos\theta}{r}\right) + \frac{1}{r^2\sin\theta}\left(-Cr\sin^2\theta - \frac{D\sin^2\theta}{r^2}\right)$$

$$= \frac{C\cos\theta}{r} + \frac{2D\cos\theta}{r^4} - \frac{C\cos\theta}{r} - \frac{2D\cos\theta}{r^4}$$

$$= 0$$

Question 4(b)

We know that in the presence of a uniform field, the charges will polarize towards the top and bottom of the sphere.



$V = 0$ at the x-y plane

$$V\left(r, \frac{\pi}{2}\right) = 0, \qquad (1), \qquad r \gg R$$

$$V(R,\theta) = \frac{Q}{4\pi\epsilon_0 R}, \qquad (2)$$

$$V(r,\theta) = -E_0\cos\theta, \qquad (3), \qquad r \gg R$$

We assume the solution

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

$$V(R,\theta) = \sum_{l=0}^{\infty}\left(A_l R^l + \frac{B_l}{R^{l+1}}\right)P_l(\cos\theta) = \left(A_0 + \frac{B_0}{R}\right) + \sum_{l=1}^{\infty}\left(A_l R^l + \frac{B_l}{R^{l+1}}\right)P_l(\cos\theta) = \frac{Q}{4\pi\epsilon_0 R}$$

$$\therefore B_0 = \frac{Q}{4\pi\epsilon_0}, \qquad A_0 = 0, \qquad A_l R^{2l+1} = B_l \text{ for } n \geq 1$$

At $r \gg R$,

$$V(r,\theta) = \sum_{l=1}^{\infty} A_l r^l P_l(\cos\theta) = A_1 r\cos\theta + \sum_{l=2}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta$$

$$A_1 = -E_0, \qquad B_1 = -E_0 R^3, \qquad A_{l>1} = B_{l>1} = 0$$

$$\therefore V(r,\theta) = \left(A_0 + \frac{B_0}{r}\right) + \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta = \frac{Q}{4\pi\epsilon_0 r} - E_0\left(r + \frac{R^3}{r^2}\right)\cos\theta$$

Question 4(c)

$$V(R,\theta) = \frac{V_0}{2}(1 + 2\cos\theta + 3\cos^2\theta), \qquad (1)$$
$$V = 0, \qquad r \gg R, \qquad (2)$$

We assume the solution

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

because $B_l = 0$ for all l, if not the potential will blow up at the origin.

At $r = R$,

$$V(R,\theta) = A_0 + A_1 R\cos\theta + A_2 R^2\left(\frac{3}{2}\cos^2\theta - 1\right) + \sum_{l=3}^{\infty} A_l R^l P_l(\cos\theta)$$
$$= \frac{V_0}{2} + V_0\cos\theta + \frac{3}{2}V_0\cos^2\theta$$

$$A_1 = \frac{V_0}{R}, \qquad A_2 = \frac{V_0}{R^2}, \qquad A_{l>2} = 0$$
$$A_0 - A_2 R^2 = \frac{V_0}{2} \quad \Rightarrow \quad A_0 = \frac{3}{2}V_0$$

$$\therefore V(r,\theta) = \frac{3}{2}V_0 + V_0\frac{r}{R}\cos\theta + V_0\frac{r^2}{R^2}\left(\frac{3}{2}\cos^2\theta - 1\right) = V_0\left[\frac{3}{2}\left(1 - \frac{r^2}{R^2}\right) + \frac{r}{R}\cos\theta + \frac{3r^2}{2R^2}\cos^2\theta\right]$$

Solutions provided by: **John Soo**
Kindly contact Physoc if you find any mistakes and etc. Thank you! ☺